Optimal Monetary Policy with Endogenous Productivity in a Small Open Economy

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Preliminary Draft

September 2018

Abstract

This paper derives the optimal monetary policy in a small open economy with endogenous productivity. The optimal policy is a targeting rule of inflation, output gap, and the terms of trade, which generates trade-off between the international purchasing power and the cost of importing R&D. Under a positive technology shock, an expansionary monetary policy, which leads to depreciation, speeds up the convergence of the technology level via a decline in R&D investment. To take advantage of this mechanism, central banks have an incentive to adjust the interest rate more aggressively. Quantitatively, the variation of the optimal monetary policy is three times larger than the standard Taylor rule and two times larger than the optimal monetary policy under an exogenous productivity process. The optimal monetary policy can improve welfare by 0.52% compared with the standard Taylor rule.

1 Introduction

In addition to fluctuations in the domestic output gap and inflation, literature has emphasized the importance for central banks in an open economy to also respond to the fluctuations in the international relative price.[1] Features that differentiate the optimal monetary policy in an open economy from that in a closed economy include the presence of home bias, local currency pricing, and the incomplete asset market. Besides those concerns, an additional

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[1] See, e.g., Corsetti and Pesenti (2001), Benigno and Benigno (2003), Corsetti and Pesenti (2005), De Paoli (2009a), Faia and Monacelli (2008), and Corsetti et al. (2010).

external distortion comes from R&D investment that may be affected by the real exchange rate, potentially through innovation that requires foreign input or willingness to invest that is decreasing in firms' export competitiveness.[2] Since productivity is often influenced by R&D investment,[3] endogenous R&D thus affects the trade-off between the terms of trade and the current as well as future output gap.

This paper aims to study the new external distortion of the real exchange rate via technology and R&D investment by asking the following questions: Given that the technology level is affected by the international relative price via R&D investment, how should central banks stabilize the economy? What is the difference between the dynamics of aggregate variables, such as output, inflation, and technology, under the three scenarios: optimal monetary policy with endogenous productivity process, the optimal monetary policy with exogenous productivity process, and the standard Taylor rule? How large is the welfare improvement when central banks implement the optimal monetary policy?

I study the optimal monetary policy and the dynamics in a small open economy model, where the productivity process can be affected by firms' endogenous choices on R&D investment. The model features four sectors: households, firms, the government, and the central bank. Households gain positive utility from consumption and negative utility from providing labor. Firms demand labor, adjust price subject to Calvo-type price stickiness, and decide whether to invest in R&D, which increases future technology. The decision to invest R&D depends on the relative size of the marginal benefit from future profit and the innovation cost denominated in the aggregate price level, which is increasing in the foreign price. The government subsidizes labor to eliminate the markup, while the central bank implements monetary policies. The asset market is assumed to be complete, and the law of one price holds. Equilibrium R&D is decreasing in real depreciation, which raises the innovation cost. The R&D channel thus generates a new terms of trade externality other than the mechanism of international purchasing power.

With the presence of the new terms of trade externality on technology, I show that the optimal monetary policy is a historically dependent targeting rule of domestic inflation, output gap, and the terms of trade gap, where the terms of trade externality becomes more sensitive to the shock on technology compared with the case with an exogenous productivity process. For example, a positive technology shock creates negative output gaps, whose duration depends on the persistence of the technology process. Central banks that decline the interest rate in order to boost domestic demand will generate depreciation. The fact

[2] See, e.g., Funk (2003), Zietz and Fayissa (1994), Becker and Pain (2008), Tabrizy (2016), Chen (2017), and Chi (2018).

[3] See, e.g., Aw et al. (2011), Doraszelski and Jaumandreu (2013), Bøler et al. (2015)Hall (2015), Reifschneider et al. (2015), Huber (2018), Anzoategui et al. (2019)

that depreciation shrinks R&D investment converges the positively deviated technology at a faster speed. Consequently, the central bank has an incentive to magnify depreciation via a more expansionary monetary policy to exploit the channel of R&D.

Moreover, associated output gap and inflation are of opposite signs under the optimal monetary policy in a model with endogenous productivity compared with cases under the domestic inflation-based Taylor rule and the optimal monetary policy without R&D. Specifically, with endogenous productivity, the output gap and inflation are negative in the initial period under a positive technology shock because the optimal monetary policy is too expansionary such that the output gap turns positive as demand sharply increases. Quantitatively, the response of the optimal monetary policy with endogenous productivity is three times larger than the domestic inflation-based Taylor rule.

Next, I quantitatively compare the welfare under the domestic inflation-based Taylor rule and the optimal monetary policy. Under the technology and foreign output shocks, I show that consumption-equivalent utility is 0.41% to 0.68% higher when implementing the optimal monetary policy. The welfare loss is decreasing in the effect of R&D on productivity level because it speeds up the convergence and thus lowers fluctuations. Moreover, the welfare loss is increasing in the elasticity of substitution between varieties as it magnifies the welfare loss from inflation variability. The quantitative result implies that endogenous productivity may be an important component that affects trade-offs among the terms of trade, the output gap, and domestic inflation.

This paper is related to the wide literature that focuses on the optimal monetary policy in an open economy. The relevance of the relative international prices depends on the assumptions of the model.[4] Some studies emphasize that optimal monetary policies in the closed and open economies are isomorphic or quantitatively similar under certain conditions (See, e.g., Clarida et al. (2001), Clarida et al. (2002), Gali and Monacelli (2005), and Benigno and Benigno (2006)). The distortion regarding terms of trade does not matter under the following assumptions: (1) The asset market is complete, (2) prices follow producer currency pricing, (3) the degree of countries' openness approaches zero.[5] In this scenario, the optimal policy should fully stabilize the domestic price and output.

Some papers have focused on more general cases where the optimal policy involves the dynamics of the terms of trades. Corsetti et al. (2009) relax the first assumption by assuming that households only borrow via an international bond. The incomplete market leads to misalignment in the international relative prices and demand imbalances. De Paoli (2009b)

[4]See Corsetti et al. (2010) for a detailed review.

[5]The third assumption can be replaced by assumptions where the substitutability between domestic goods and foreign goods, the inverse of the intertemporal elasticity, and the substitutability between goods produced in different foreign countries all equal one.

shows that under incomplete markets, the optimal monetary policy stabilizes the exchange rate when the home and foreign goods are not close substitutes. Devereux and Engel (2003), Corsetti and Pesenti (2005), and Engel (2011) relax the second assumption by assuming local currency pricing (LCP). Devereux and Engel (2003) show that optimal monetary policy keeps the exchange rate fixed under LCP when the policy simultaneously removes the currency misalignment and keeps the world aggregate demand at its efficient level. Engel (2011) shows that they are separate objects and that the optimal monetary policy concerns consumer price inflation as central banks not only concern the price dispersion in the domestic market but also the foreign market. De Paoli (2009a) relaxes the third assumptions and derive the optimal monetary policy via the linear-quadratic approach. She shows that the optimal monetary policy involves the terms of trade gap, where the volatility of the exchange rate crucially depends on the substitutability between domestic and foreign goods. Benigno and Benigno (2006) further highlight the potential gains of welfare under policy coordination that involves the terms of trade and the foreign output gap. I contribute to the literature by providing a new channel through which endogenous technology affects the trade-off among the terms of trade, output gap, and domestic inflation. Specifically, I show that with the presence of this new distortion, stabilizing domestic inflation is not optimal even when all assumptions (1), (2), and (3) hold.

This paper is also related to recent works that consider the monetary policy in the model of endogenous productivity. For example, Annicchiarico and Pelloni (2016) focus on the optimal monetary policy under a New Keynesian model with endogenous productivity from the creation of patented technologies, similar to the work of Comin and Gertler (2006). In their model, innovation can enhance monopoly power and thus should be heavily invested during booms. Accordingly, the optimal monetary policy should deviate from strict inflation targeting and be counter-cyclical. Compared with the policy in the framework with exogenous productivity, the relative magnitude of the policy depends on the type of shock.

Garga and Singh (2018) focus on the fall of endogenous investment and permanent output loss (also known as output hysteresis) caused by the failure of stabilizing the economy under the standard Taylor-type monetary policy. They show that the presence of output hysteresis depends on both the monetary policy and the binding condition of the zero lower bound. In this framework, the optimal monetary policy can be closely approximated by a hysteresis targeting rule, which takes the endogenous productivity into account. The endogenous technology growth is modeled by the Schumpeterian growth model, similar to Howitt and Aghion (1998). However, technology improvement features a "step size" effect where the growth rate of the technology remains the same after innovation. This does not match the current observation that technology growth can be decreasing in the technology level.

The main difference between this paper and the works of Annicchiarico and Pelloni (2016) and Garga and Singh (2018) is that I consider an open economy, whereas both papers focus on a closed economy setting and thus can not speak to the terms of trade externality. The ignorance of the relationship between R&D and the terms of trade may lead to welfare loss, depending on the persistence of the productivity process.

2 A Model with Endogenous Productivity

The model closely follows Chi (2018), who extends the works of Gali and Monacelli (2005) by adding R&D investment which can be endogenously determined by firms. The model consists of four sectors: the household, a continuum of firms, the central bank and the government. The Household demands domestic and foreign variety of goods and provides labor hours. The central bank selects the monetary policy and the government subsidizes wage to remove the steady-state mark-up.

Firms decide whether to invest in R&D by comparing the marginal gain from future profit and the fixed cost denominated in the CPI. They then adjust prices subject to Calvo-type price stickiness and demand labors. For tractability, firms sign a joint agreement regarding the profit allocation where aggregate profit are allocated between two groups of firms: firms that conduct R&D and firms that do not. Firms in the group of conducting R&D will secure a higher profit, and the profit is uniformly distributed across firms in each group. Together with the assumption that technology level is publicly observable, each firm's technology follows the same technology process, where R&D is determined by aggregate variables, such as the terms of trade, as well as the dispersion of the firm-specific innovation cost.

To highlight the external distortion results from the R&D channel, I assume that the asset market is complete and that firms follow producer currency pricing. The economy is assumed to be small such that foreign output and prices are taken as given.

2.1 Household

The household maximizes the utility:

$$max E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi}$$

subject to the budget constraint

$$\int_0^1 P_{H,t}(j)C_{H.t}(j)dj + \int_0^1\int_0^1 P_{i,t}(j)C_{i,t}(j)djdi + E_t\{Q_{t,t+1}D_{t+1}\} \le D_t + W_tN_t + T_t,$$

where σ is the intertemporal elasticity of substitution; W_t is the nominal wage; $Q_{t,t+1}$ is the stochastic discount factor; D_{t+1} is the nominal payoff in period $t+1$ of the portfolio in period t; $C_{i,t}$ is a composite of all the goods produced in country i and is given by

$$C_{i,t} = (\int_0^1 C_{i,t}(j)^{(\epsilon-1)/\epsilon}dj)^{\epsilon/(\epsilon-1)},$$

where ϵ is the elasticity of substitution between the variety of domestic goods. The consumption composite C_t is given by

$$C_t = [(1-\alpha)^{1/\eta}(C_{H,t})^{(\eta-1)/\eta} + \alpha^{1/\eta}(C_{F,t})^{(\eta-1)/\eta}]^{\eta/(\eta-1)},$$

where η is the elasticity of substitution between home and foreign goods. α captures the openness. $C_{F,t}$ is the foreign consumption composite and is given by

$$C_{F,t} = (\int_0^1 C_{i,t}^{\gamma/(\gamma-1)}di)^{\gamma/(\gamma-1)},$$

where γ is the elasticity of substitution between goods from different countries; Under first-order Taylor approximation, the optimality can be characterized by

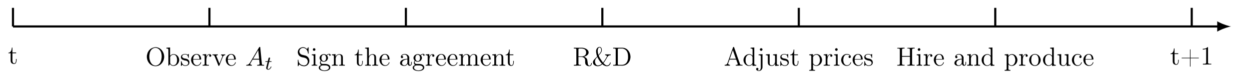
$$w_t - p_t = \sigma c_t + \psi n_t, \tag{1}$$

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}(i_t - E_t\pi_{t+1} + log\beta), \tag{2}$$

where $i_t \equiv -log(E_t(Q_{t,t+1}))$. Equation (1) is the labor supply and equation (2) is the aggregate demand that equates the marginal utility of saving and consumption.

2.2 Firm

There is a continuum of monopolistic firms, which invest in R&D subject to a heterogeneous cost, adjust prices and demand labor. The timing of the firm decision is as follows:



2.2.1 Profit agreement and R&D decision

For tractability, aggregate technology and the R&D level are publicly observable. This ensures that firms face the same technology process and prevents a divergence of firm-specific technology when firms that have higher initial technology gain a higher marginal benefit of further investing in R&D. The technology process is given by

$$ln(A_t(j)) = a_t(j) = exp(\alpha_0 + \alpha_1 a_{t-1} + \alpha_2 d_{t-1}), \tag{3}$$

where $d_{t-1} = \int d_{t-1}(j)dj$ is the share of firms that decide to invest in R&D in period $t-1$, and $d_{t-1}(j)$ equals one if firm j decides to invest in R&D and zero otherwise. Current technology is affected by past technology and the past share of firms investing in R&D. α_1 is the persistence of the technology process, and α_2 measures the extent to which R&D boosts aggregate technology. Following Chi (2018), firms sign an agreement regarding the rule of the distribution of the aggregate profit before producing output. The expected profit that firms can be distributed depends on their current choice on investing in R&D. To provide an incentive for firms to invest, firms that invest in R&D will secure a higher profit than those do not. This agreement can also insure the uncertainty of price stickiness through the aggregation of individual profit. $\tilde{\Pi}_{t+1}(j)$ denotes the after-allocation profit that firm j will receive after selling the production in period $t+1$. The profit agreement is given by

$$\tilde{\Pi}_{t+1}(j) = \begin{cases} \Phi_{t+1} P_{H,t+1}^{1/(1-\alpha_p)} (e^{\alpha_0+\alpha_1 a_t+\alpha_2})^{1/(1-\alpha_p)} X & if\ d_t(j) = 1 \\ \Phi_{t+1} P_{H,t+1}^{1/(1-\alpha_p)} (e^{\alpha_0+\alpha_1 a_t})^{1/(1-\alpha_p)} X & if\ d_t(j) = 0 \end{cases}, \tag{4}$$

where X normalized profits to ensure that the sum of the before-allocation profit is equal to the sum of the after-allocation profit, and $\Phi_t \triangleq [(1-\tau)W_t]^{\alpha_p/(\alpha_p-1)} (\alpha_p^{\alpha_p/(1-\alpha_p)} - \alpha_p^{1/(1-\alpha_p)})$ is a common factor across all firms. The decision on R&D investment is depends on the size of the margin gain from the future allocated profit and the firm-specific innovation cost $ce^{-\mu_t(j)}$. c is a fixed cost, and $e^{-\mu_t(j)}$ is a random scale where $\mu_t(j) \sim u(ln\epsilon_d, 0)$ and $\epsilon_d \in (0,1)$. The decision on R&D investment can be characterized as follows:

$$d_t(j) = \begin{cases} 1 & if\ E_t\hat{\Pi}_t(j; d_t = 1) - E_t\hat{\Pi}_t(j; d_t = 0) \geq ce^{-\mu_t(j)} P_t A_t \\ 0 & otherwise \end{cases}, \tag{5}$$

where

$$P_t = [(1-\alpha)P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta}]^{1/(1-\eta)},$$

$$\hat{\Pi}_t(j; d_t) = \sum_{s=t+1}^{\infty} \beta_F^s \tilde{\Pi}_s(j; d_t).$$

$\hat{\Pi}_t(j; d_t)$ denotes firm j's discounted sum of the profit stream. P_t is the consumer price index, and α is a measure of openness. The fact that the nominal cost of R&D is affected by the CPI implies that the R&D decision is decreasing in the terms of trade. This is the key mechanism where the terms of trade affect the productivity process and future output. The feature that the cost of R&D is increasing in A_t captures the "stepping-on-toe" effect where technology becomes more expensive as the technology level rises (See, e.g., Aghion et al. (2009)).

Following Chi (2018), equilibrium R&D investment is a function of aggregate variables:

$$d_t = f(a_t, S_t, E_t y_{t+1}^*, E_t \pi_{H,t+1}, E_t S_{t+1}), \tag{6}$$

where $\partial d_t / \partial S_t < 0$ when α_2 is not extremely low.

2.2.2 Labor demand and production

Note that the profit agreement ensures that the after-allocation profit is proportional to the before allocation profit, implying that the optimization and the price setting are identical to the case without the profit agreement. The standard firms' optimization problem is given by

$$\max_{P_{H,t}(j), Y_t(j), N_t(j)} \Pi_t(j) = P_{H,t}(j) Y_t(j) - (1-\tau) W_t N_t(j),$$

where τ is the wage subsidy that corrects the mark-up. The production function is given by

$$Y_t(j) = A_t(j) N_t(j)^{\alpha_p}.$$

The optimality can be characterized by the labor demand:

$$\alpha_p N_t(j)^{\alpha_p - 1} P_{H,t}(j) A_t(j) = (1-\tau) W_t. \tag{7}$$

The associated profit function can therefore be given by

$$\Pi_t(j) = \Phi_t[P_{H,t}(j)A_t(j)]^{1/(1-\alpha_p)}. \tag{8}$$

By comparing equation (4) and (8), we confirm that the after-allocation profit is proportional to the before allocation profit, regardless of firms' decision on R&D.

2.2.3 Price setting

As shown in the literature (See, e.g., Gali and Monacelli (2005)), the optimal price setting under homogeneous technology across firms is given by

$$p^*_{H,t} = \mu + (1-\beta\theta)\Sigma_{k=0}^{\infty}(\beta\theta)^k E_t[\tilde{mc}_{t+k|t} + p_{H,t+k}]$$

up to the first-order approximation, where $p^*_{H,t}$ is the log of the newly set domestic price, $\mu = -\tilde{mc} = log(\mathcal{M})$ is the log of the steady-state mark-up, and θ is the probability that firms can adjust price.

2.3 Equilibrium

Following Gali and Monacelli (2005) and Chi (2018), the log-linearized equilibrium can be jointly characterized by the Phillips curves and the aggregate demand, which are given by

$$\tilde{y}^n_t = E_t\tilde{y}^n_{t+1} - \frac{1}{\sigma_\alpha}(i_t - E_t\pi_{H,t+1} - r^n_t),$$

$$\pi_{H,t} = \beta E_t\pi_{H,t+1} + K\tilde{y}_t,$$

where

$$K = \frac{(1-\beta\theta)(1-\theta)}{\theta}\left(\frac{\psi}{\alpha_p} - \frac{\alpha_p - 1}{\alpha_p}\right),$$

$$r^n_t = \rho + \frac{\zeta\sigma_\alpha\psi}{\sigma_\alpha+\psi}E_t(\Delta y^*_{t+1}) - \sigma_\alpha\Gamma_\alpha(-\alpha_0 + (1-\alpha_1)a_t - \alpha_2 d_t),$$

$$y^n_t = \mathit{\Gamma}_0 + \Gamma_a a_t + \Gamma_\star y^*_t, \tag{9}$$

where $\sigma_\alpha = \sigma/[1+\alpha(\sigma\gamma + (1-\alpha)(\sigma\eta - 1) - 1)]$, $\Gamma_a = (1+\psi)(1+\psi-\alpha_p(1-\sigma_\alpha))^{-1} > 0$, and $\tilde{y}^n_t = y_t - y^n_t$ denotes the output gap. The natural rate of interest is increasing in R&D because current investment raises future output and consumption, which can only be accommodated with a higher natural rate of interest.

3 Optimal Monetary Policy with Endogenous R&D

With the analytical model in hand, I first approximate the utility-based loss function via the linear-quadratic approach and derive the optimal monetary policy as a targeting rule of inflation, output gap, and the terms of trade against the technology shock and the foreign output shock. I then compare the dynamics between the optimal monetary policy with endogenous as well as exogenous productivity, and the domestic inflation-based Taylor rule. Finally, I estimate the quantitative welfare improvement under different values of the effect of R&D on technology and the markup.

3.1 Optimal Monetary Policy

Similar to the optimal policies involving the external distortion studied in the literature, central banks implement an optimal monetary policy to trade-off among misalignment of the terms of trade, the output gap, and inflation. However, the endogenous productivity process complicates the optimality problem as the current terms of trade as well as current domestic inflation affect current R&D investment and thus influences the technology level in all future periods, depending on the persistence of the technology process. Central banks thus trade-off the misalignment of the terms of trade not only with the current but also the future output gap and inflation. Throughout the analysis of the optimal monetary policy, I assume that the central banks can commit to the policy. Proposition 1 characterizes the optimal monetary policy.

Proposition 1. *The optimal monetary policy is given by*

$$\mathcal{C}_y\tilde{y}_t + \mathcal{C}_s\tilde{s}_t + \mathcal{C}_\pi\hat{\pi}_{H,t} = \mathcal{C}^0_y\tilde{y}_{t-1} + \mathcal{C}^0_s\tilde{s}_{t-1} + \mathcal{C}^0_\pi\hat{\pi}_{H,t-1} + \mathcal{C}e_t,$$

where $\mathcal{C}$ denotes constants, and $\tilde{x}_t = \hat{x}_t - \hat{x}^T_t$ and $\hat{x}_t = ln(X_t/\bar{X})$ for any given variable x. x^T denotes the target value which minimizes the loss function when domestic inflation equals zero. The targeting gap $\hat{x}^T_t = q^e_x e_t$ is a function of a set of shock e_t, where $e_t = [\hat{a}_t, \hat{y}^*_t]'$ includes the technology shock and the foreign output shock.

Proof: See Appendix 5.1.

The optimal monetary can also be written in terms of growth rates of variables:

$$\mathcal{C}_y(\tilde{y}_t - \frac{\mathcal{C}^0_y}{\mathcal{C}_y}\tilde{y}_{t-1}) + \mathcal{C}_s(\tilde{s}_t - \frac{\mathcal{C}^0_s}{\mathcal{C}_s}\tilde{s}_{t-1}) + \mathcal{C}_\pi\hat{\pi}_{H,t} - \mathcal{C}^0_\pi\hat{\pi}_{H,t-1} = \mathcal{C}_a\hat{a}_t + \mathcal{C}_{y^*}\hat{y}^*_t.$$

The equation characterizes a historically dependent targeting rule of domestic inflation, output gap, and the terms of trade gap. The target values of the three variables are determined by the deviation of technology and foreign output, both are determinants that pin down the level of R&D and thus the path of real variables. The deviation of terms of trade and its coefficient capture two effects: (1) The trade-off related to the international purchasing power, where depreciation leads to higher foreign demand for domestic goods and appreciation may increase welfare via higher domestic consumption of foreign goods,[6] (2) The effect of the terms of trade on R&D investment and future technology.

To analyze the trade-offs among the three targeting gaps, Figure 1 plots the coefficients considering a range of parameter α_2, which is the key variable of that determines the effect of R&D investment on the technology process. The selected range of α_2 includes the empirical value 0.044. The lowest selected value $\alpha_2 = 0$ corresponds to the case where the terms of trade affects central banks' stabilization only through the effects of relative international prices on foreign demand of domestic goods and domestic consumption of foreign goods (e.g. De Paoli (2009a)). In this scenario, central banks stabilize a positive technology shock by allowing domestic deflation, an output gap that is smaller than the target level $q_y^e[1,1]\hat{a}_t$, and depreciation milder than $q_s^e[1,1]\hat{a}_t$. Depreciation implies that welfare improvement from higher foreign demand of domestic goods outweighs the welfare deterioration results from the lower purchasing power of domestic households.

The dynamic of the real exchange rate can be further affected by the channel of endogenous R&D investment. When α slightly increases from zero, since $\mathcal{C}_a$ turns positive and $\mathcal{C}_s$ remains positive, central banks respond to a positive technology shock by depreciation that is more drastic than $q_s^e[1,1]\hat{a}_t$. The intuition is that real depreciation discourage firms to invest in R&D as it becomes costly. Consequentially, the positive deviation of technology will converge at a faster speed compared with the case without huge depreciation. Such effect lowers the welfare loss by shrinking fluctuations in future periods. With positive α_2, the magnitude of depreciation in response to a positive technology shock is always larger than the case where $\alpha_2 = 0$.

3.2 The Effect of a Technology Shock

This section compares the dynamic responses of three alternatives: the optimal monetary policy under the exogenous as well as endogenous productivity process, and the domestic inflation-based Taylor rule in the model with endogenous productivity, which is given by

[6] De Paoli (2009a) shows that depreciation is welfare improving when foreign and domestic goods are not close substitutes.

$$i_t = \rho + \phi_\pi \pi_{H,t} + \phi_y \hat{y}_t.$$

Throughout the quantitative exercise, I follow the coefficients estimated by Chi (2018), as shown in Table 1. He calibrates the model to match the empirical moments using the data from Taiwan, which is often viewed as a small open economy. The paper also estimates the technology process (3) using plant-level data of manufacturing industry in Taiwan.

Figure 2 and 3 plot the impulse responses under the one-percent positive technology shock. Technology in the exogenous productivity model follows a standard AR(1) process, as shown by the dash-dotted line. The positive technology shock leads to higher output and consumption in all models. However, the interest rate responds more in the case of the optimal monetary policy under endogenous productivity process. The deviation of optimal rate is three times as large as the deviation of standard Taylor rule, and is two times as large as the optimal monetary policy under exogenous productivity.

The intuition behind the more expansionary nominal rate is that it lowers R&D innovation through two channels: (1) real depreciation, and (2) future inflation, which is triggered by a positive output gap. The future inflation will lower R&D if the output elasticity of labor α_p is large enough such that effect of the increasing future wage dominates the effect of the increasing profit. Therefore, depreciation leads to a decline in the R&D level, which offsets the technology shock and thus shortens the persistence of the deviation. As shown in Figure 2, the optimal monetary policy under endogenous R&D suggests a negative deviation of R&D investment and a shorter persistence of the technology shock. Note that the Taylor rule suffers from the longest shock duration because it instead suggests an increase in R&D even the interest rate decreases and the exchange rate depreciates. The main reason that drives the increase in R&D is the deflation that lowers the nominal cost of R&D in the current period. With shorter persistence of the shock, consumption, output and the real exchange rate under the optimal monetary policy with endogenous productivity also converge back to the steady state at a faster speed.

Another important difference between the optimal monetary policy with endogenous productivity and the other cases is the dynamics of the output gap. The output gap jumps up in the initial period when productivity is endogenous while the output gap is negative in the other two cases. Under the case without R&D, central banks partly offset the negative output gap by lowering the nominal interest rate. The negative output gap then leads to domestic deflation as the increase in goods supply is larger than goods demand. The dynamics of the terms of trade and the real exchange rate is determined by two forces :

(1) nominal depreciation results from the decrease in the interest rate via the UIP, and (2) the deflation of domestic price due to the negative output gap. Both effects lead to real depreciation and deterioration of the terms of trade.

However, to exploit the R&D channel that speeds up the convergence of technology, central banks in the model of endogenous productivity optimally lowers the R&D level by further dampening the interest rate. Demand of goods then rises to a point where the output gap turns positive and the domestic price inflates. The terms of trade is now determined by two offsetting forces: the nominal depreciation from UIP and the domestic deflation that results from the positive output gap. Quantitatively, the first effect outweighs the second and leads to larger real depreciation compared with the previous two cases. [7] The above mechanism generates a negative deviation of R&D investment. In period 2, since the technology level has dropped, the interest rate converges and domestic demand increases by less, resulting in a negative output gap and subsequent domestic deflation. Compared with the inflation-based Taylor rule, central bankers improve welfare by enduring a large deviation of output gap in period 1 in exchange for lower deviations of all future output gaps. Moreover, the optimal monetary policy improves welfare by featuring smaller deflation in all future periods since the associated output gap converges more quickly.

Special case: $\sigma = \gamma = \eta = 1$

To highlight the new terms of trade externality, I will revisit the special case, where the solution of closed and open economies are isomorphic without endogenous productivity process. According to Gali and Monacelli (2005), this can be achieved by setting the substitutability between domestic goods and foreign goods, the inverse of the intertemporal elasticity, and the substitutability between goods produced in different foreign countries to be one.

Figure 4 and 5 plot the impulse responses under a one-percent technology shock. The key finding is that the stabilization of the domestic inflation is no longer optimal in this special case, contradicting results in the literature. Specifically, both the output gap and domestic inflation feature a drop in the initial period, and subsequently switch to positive inflation in the next period. Compared with the previous values of σ, γ, and η listed in Table 1, the main difference comes from the sharp increase in the inverse of the intertemporal elasticity σ from 0.3416 to 1. Such change lowers households' willingness to save when the real interest rate rises, and thus lowers the value of the Arrow-Debreu asset. Given that assets can be traded internationally, a lower value of the Arrow-Debreu asset declines the correlation between consumption and the real exchange rate.

[7] Under the optimal monetary policy with endogenous productivity, real depreciation in the initial period is 3.56% higher than the case of the optimal monetary policy with exogenous productivity.

Under a technology shock, the central bank implements expansionary policy that lowers the nominal interest rate, leading to real depreciation. However, the increase in domestic demand resulting from depreciation is lower than the magnitude in the previous model. This can be observed from a lower increase in consumption and output in Figure 5 than in Figure 3. As a result, the expansionary policy is unable to close the negative output gap, which generates a domestic deflation. The mechanism of R&D then starts to kick in from period 1. Because of a decrease in R&D in the initial period, technology quickly converges back such that the domestic supply declines. Such effect is too strong so that the output gap and inflation turns positive.

3.3 Welfare Analysis

To evaluate the importance of the optimal monetary policy, the welfare analysis is simulated and measured in terms of consumption-equivalent utility relative to the efficient steady state under flexible prices and zero mark-up. To derive the subsidy that eliminates the mark-up, I will first use the optimality condition of the household where the real wage is given by

$$W_t/P_t = \frac{\alpha_p(1-\alpha)}{(1-\alpha+\alpha\omega)}. \tag{10}$$

By assuming that the market is complete, the domestic consumption is proportional to the foreign consumption via the terms of trade, and the correlation is determined by the intertemporal elasticity and the substitution and the degree of openness. Specifically, the demand of the home country and the rest of the world is correlated:

$$c_t = c_t^* + \frac{1-\alpha}{\sigma} s_t. \tag{11}$$

With equation (10) and (11), the wage subsidy is given by

$$ln(1-\tau) = ln(1-\frac{1}{\epsilon}) + \bar{a} + \alpha\bar{s} - ln(\frac{1-\alpha}{1-\alpha+\alpha\omega}\alpha_p) + ln\alpha_p, \tag{12}$$

where $\bar{x}$ denotes the log of x in the efficient steady-state, which can be solved by equation (6), (3), (11), and (9). By applying the parameterization in Table (1), the efficient wage subsidy equals 1.18%. See Appendix (5.1.1) for the quadratic-linearized function of welfare loss.

Table 2 reports the welfare loss under the domestic inflation-based Taylor rule and the optimal monetary policy under endogenous technology. There exist two shocks: the technology and the world output shock, whose standard deviations are assumed to be 0.1%. The welfare loss is decreasing in the level of the steady state mark-up (i.e. increasing in the elasticity of substitution) due to the fact that the degree of penalization from inflation variability is increasing in ϵ, as shown in equation (21). On the other hand, the welfare loss is decreasing in the coefficient of R&D because R&D shortens the persistence and the magnitude of shocks. Within the selected values of α_2 and ϵ, the optimal monetary policy can increase the welfare by 0.41 to 0.68%, implying that the channel of endogenous R&D is quantitatively important when designing the optimal monetary policy.

4 Conclusion

This paper studies the optimal monetary policy in a model with endogenous productivity, which is increasing in the level of R&D. I show that when R&D is affected by other aggregate variables such as the real exchange, endogenous productivity generates an additional trade-off between the terms of trade gap and the output gap. This channel can be more relevant when the technology process becomes more persistent. Specifically, I focus on the mechanism where R&D is decreasing in real depreciation by assuming that technology innovation requires input from foreign countries.

The optimal monetary policy is a historically dependent targeting rule of output gap, domestic inflation, and the gap of the terms of trade. Compared with the case under exogenous productivity, the optimal monetary policy under endogenous productivity suggests the gap of the terms of trade be more sensitive to the technology shock. When facing a technology shock, the impulse response of the nominal interest rate under the optimal monetary policy with endogenous productivity is three times higher than the case under the domestic inflation-based Taylor rule and two times higher than the optimal monetary policy with exogenous productivity. The intuition is that the central bank, which faces a positive technology shock, exploits the channel of endogenous productivity by implementing a more expansionary policy, which leads to real depreciation and lower R&D. Lowering R&D then speeds up the convergence of the technology and thus lowers fluctuations of aggregate variables.

Moreover, associated output gap and inflation are of opposite signs compared with the other two scenarios. Specifically, with endogenous productivity, the output gap and inflation are negative in the initial period under a positive technology shock because the optimal monetary policy is too expansionary so that the output gap turns positive as demand sharply increases.

Finally, I show that in a small open economy model that features endogenous productivity, implementing the optimal monetary policy can increase the consumption-equivalent welfare by 0.41 to 0.68% relative to the domestic inflation-based Taylor rule.

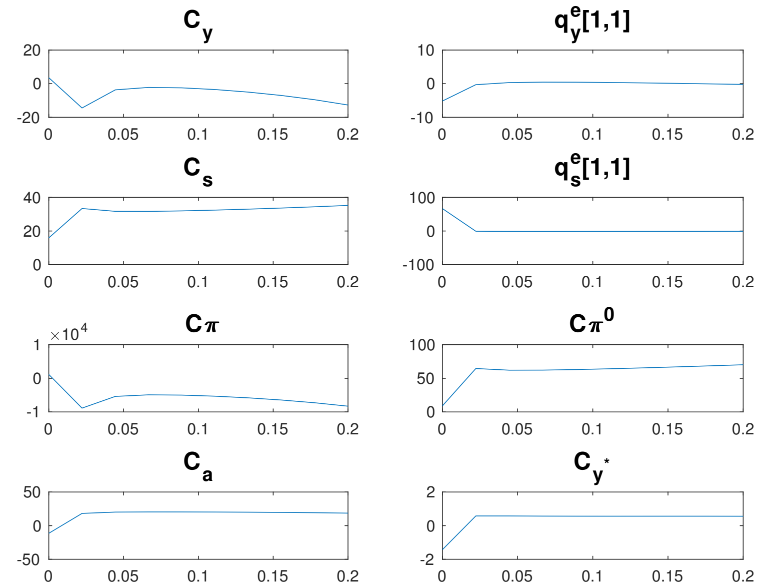


Figure 1: Coefficients of the optimal monetary policy

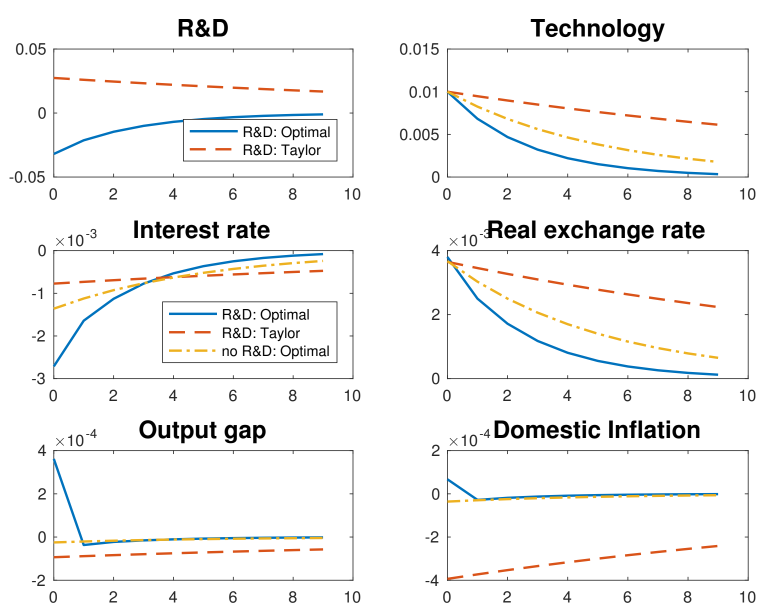


Figure 2: Impulse responses under 1% technology shock

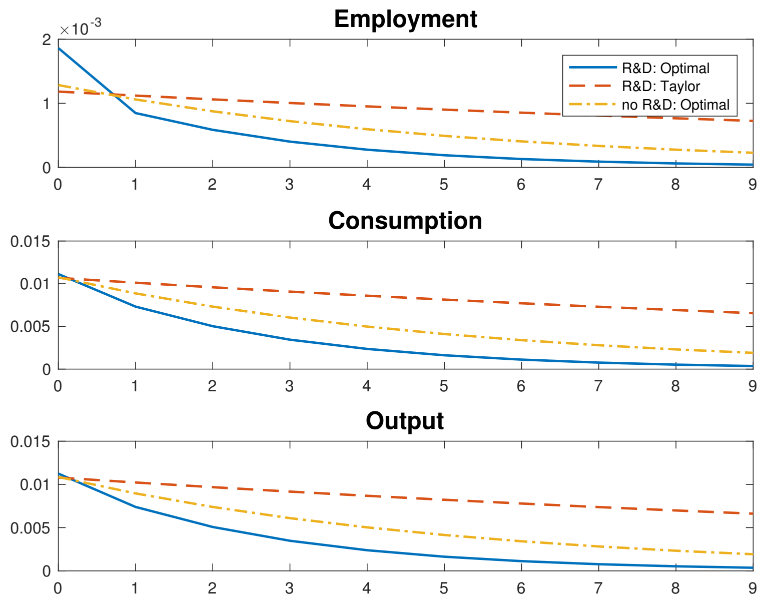


Figure 3: Impulse responses under 1% technology shock

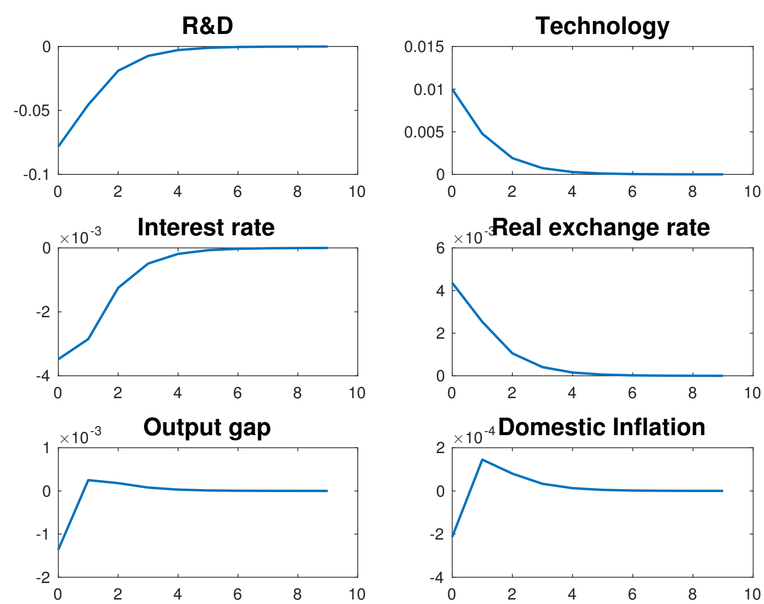


Figure 4: Impulse responses under 1% technology shock when $\sigma = \gamma = \eta = 1$

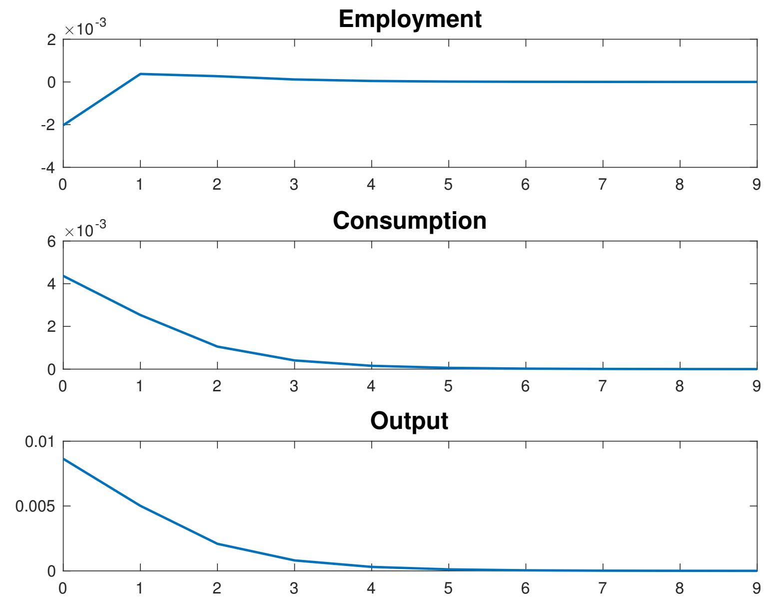


Figure 5: Impulse responses under 1% technology shock when $\sigma = \gamma = \eta = 1$

Table 1: Value of Parameters

Parameter	Value	Description	Source
η	0.89	Substitutability btw domestic and foreign goods	Feenstra et al. (2014)
γ	1.06	Substitutability btw goods from different countries	Feenstra et al. (2014)
θ	0.75	Price Stickiness	Gali and Monacelli (2005)
α_p	0.67	Output elasticity of labor	Standard value
β	0.99	Discount rate	Standard value
ψ	1.7215	Labor supply elasticity $= 1/\psi$	Chi (2018)
σ	0.3416	Intertemporal elasticity of substitution	Chi (2018)
α	0.4950	Degree of openness	Chi (2018)
ϵ_d	0.9729	Dispersion of R&D cost	Chi (2018)
c	1.3703	Fixed R&D cost	Chi (2018)
α_0	-0.002	Technology process: intercept	Chi (2018)
α_1	0.825	Persistence of technology	Chi (2018)
α_2	0.044	The effect of R&D on technology	Chi (2018)
ϕ_y	1.5	Taylor rule coefficient on output gap	Chi (2018)
ϕ_π	1.95	Taylor rule coefficient on inflation	Chi (2018)

Table 2: Welfare loss

	Taylor	Optimal	Dev %	Taylor	Optimal	Dev %	Taylor	Optimal	Dev %
	$\alpha_2 = 0.04$			$\alpha_2 = 0.045$			$\alpha_2 = 0.05$		
$\epsilon = 7.67$	-0.27	-0.27	0.50	-0.22	-0.21	0.41	-0.15	-0.14	0.60
$\epsilon = 6$	-0.23	-0.22	0.56	-0.17	-0.16	0.52	-0.10	-0.09	0.60
$\epsilon = 5$	-0.19	-0.18	0.65	-0.12	-0.12	0.61	-0.05	-0.04	0.68

Notes: The "Dev %" is calculated as the percentage gap of utility. The welfare loss is calculated by simulating 100,000 periods and summing up the utility loss relative to efficient steady state. $\epsilon = 7.67$, 6, *and* 5 *correspond to markup* 15%, 20%, *and* 25%.

5 Appendix

5.1 Derivation of the optimal monetary policy

5.1.1 Welfare Approximation

Steps for deriving the optimal policy mainly follows Woodford (2002), Benigno (2004) and De Paoli (2009a). The optimal policy is derived by maximizing household's welfare subject to the condition of the complete risk-sharing, the exchange rate that links domestic and foreign prices, the rule of price setting, and the aggregate demand. The social planner maximizes the following utility, which is increasing in consumption and decreasing in labor hour:

$$E_0 \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)].$$

I then approximate the objective via the linear-quadratic approach around the symmetric steady state. The approximation of the positive utility from consumption is given by

$$u(C_t) = t.i.p + u_c(\bar{C})\bar{C}[\hat{c}_t + \frac{1}{2}(1-\sigma)\hat{c}_t{}^2], \tag{13}$$

where the positive gain from consumption is of CRRA type such that $u(C) = C^{(1-\sigma)}/(1-\sigma)$. Disutility of the labor hour is also assumed to be a CRRA function where $v(N) = N^{(1+\psi)}/(1+\psi)$. Disutility from labor hour and its second order approximation are given by

$$\begin{aligned}
\frac{N_t^{1+\psi}}{1+\psi} &= \frac{1}{1+\psi}[\int_0^1 N_t(i)di]^{1+\psi} \\
&= \frac{1}{1+\psi}[\int_0^1 (\frac{Y_t(i)}{A_t})^{1/\alpha_p} di]^{1+\psi} && (14) \\
&\triangleq \frac{1}{1+\psi}[\int_0^1 V(Y_t(i), A_t)di]^{1+\psi} && (15) \\
&= \frac{\bar{N}^{1+\psi}}{1+\psi} + \int \{\bar{N}^{\psi}\frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}\hat{y}_t(i) + \frac{1}{2}[\psi\bar{N}^{\psi-1}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{2/\alpha_p} + \bar{N}^{\psi}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}]\hat{y}_t(i)^2 && (16) \\
&- [\psi\bar{N}^{\psi-1}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{2/\alpha_p} + \bar{N}^{\psi}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}]\hat{a}_t\hat{y}_t(i) - \bar{N}^{\psi}\frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}\hat{a}_t \\
&+ \frac{1}{2}[\psi\bar{N}^{\psi-1}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{2/\alpha_p} + \bar{N}^{\psi}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}]\hat{a}_t^2\}di\} + f.t && (17)
\end{aligned}$$

$$= t.i.p + \bar{N}^{\psi} V_y(\bar{Y}) \int_0^1 [\hat{y}_t(i) + \frac{1}{2}(1+\psi)(\frac{1}{\alpha_p})\hat{y}_t(i)^2 - (1+\psi)(\frac{1}{\alpha_p})\hat{a}_t\hat{y}_t(i)\}di + f.t.$$

$$= t.i.p + u_c(\bar{C})\bar{C}\frac{\bar{N}^{\psi} V_y(\bar{Y})\bar{Y}}{u_c(\bar{C})\bar{C}} \int_0^1 [\hat{y}_t(i) + \frac{1}{2}(1+\psi)(\frac{1}{\alpha_p})\hat{y}_t(i)^2 - (1+\psi)(\frac{1}{\alpha_p})\hat{a}_t\hat{y}_t(i)]di$$

$$+ f.t, \tag{18}$$

where

$$\hat{x}_t = lnX_t - ln\bar{X},$$

$$V_y(\bar{Y})\bar{Y} = \frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p},$$

$$\frac{-\bar{N}^{\psi} V_y(\bar{Y})}{u_c(\bar{C})} = \frac{\epsilon - 1}{\epsilon},$$

and $f.t.$ denotes joint partial derivatives where one variable is in the current period, and the other is in the future period. The term $t.i.p$ indicates variables that are independent of the policy. The fact that future output and welfare is affected by current R&D via the technology process complicates the maximization problem. Note that a_{t+i} is endogenous because today's variables will affect both future technology and R&D investment. $f.t.$ is thus important in the sense that it captures how current aggregate variables can affect future aggregate variables via endogenous R&D (e.g. $\tilde{y}_{t+1}\tilde{s}_t$, $\tilde{y}_{t+1}\hat{p}_{H,t}$). These effects do not exist in a model that features exogenous technology, which differentiate this paper from earlier works (e.g., De Paoli (2009a)). The approximation of $f.t.$ will be presented later.

Note that with condition that under the first-order Taylor approximation at the symmetric steady state with $\bar{Q} = 1$ and $\bar{S} = 1$, we obtain $\bar{Y} = \bar{C}$, equation (18) can then be written as

$$\frac{N_t^{1+\psi}}{1+\psi} = t.i.p - u_c(\bar{C})\bar{C}\frac{\epsilon - 1}{\epsilon} \int_0^1 [\hat{y}_t(i) + \frac{1}{2}(1+\psi)(\frac{1}{\alpha_p})\hat{y}_t(i)^2 - (1+\psi)(\frac{1}{\alpha_p})\hat{a}_t\hat{y}_t(i)]di + f.t$$

$$= t.i.p - u_c(\bar{C})\bar{C}\frac{\epsilon - 1}{\epsilon}[E_t(\hat{y}_t(i)) + \frac{1}{2}(1+\psi)(\frac{1}{\alpha_p})[E_t(\hat{y}_t(i))^2 + Var(\hat{y}_t(i))] - (1+\psi)(\frac{1}{\alpha_p})\hat{a}_t E_t(\hat{y}_t(i))]$$

$$+ f.t. \tag{19}$$

By using the linear-quadratic form of the consumption composite $Y_t = (\int_0^1 y_t(i)^{(\epsilon-1)/\epsilon} di)^{\epsilon/(\epsilon-1)}$, which is given by

$$\hat{y}_t = E_t(\hat{y}_t(i)) + \frac{1}{2}(1 - \frac{1}{\epsilon})Var(\hat{y}_t(i)),$$

equation (19) can be further written as

$$\frac{N_t^{1+\psi}}{1+\psi} = t.i.p - u_c(\bar{C})\bar{C}\frac{\epsilon-1}{\epsilon}\{\hat{y}_t - \frac{1}{2}(1-\frac{1}{\epsilon})Var(\hat{y}_t(i)) + \frac{1}{2}(1+\psi)(\frac{1}{\alpha_p})[\hat{y}_t^2 + \mathcal{O}(||\xi^3||) + Var(\hat{y}_t(i))] \tag{20}$$

$$-(1+\psi)(\frac{1}{\alpha_p})[\hat{a}_t\hat{y}_t + \mathcal{O}(||\xi^3||)]\},$$

where demand of goods is

$$logY_t(i) = logY_t - \epsilon(logP_{H,t}(i) - logP_{H,t}),$$
$$\Rightarrow Var(logY_t(i)) = Var(\hat{y}_t(i)) = \epsilon^2 Var(p_{H,t}(i)).$$

Following Woodford (2011), the discounted sum of the variance of domestic prices can be approximated as follows:

$$\sum_{t=0}^{\infty}\beta^t Var(p_{H,t}(i)) = \sum_{t=0}^{\infty}\beta^t\frac{\theta}{(1-\theta)(1-\beta\theta)}\pi_{H,t}^2 + t.i.p + \mathcal{O}(||\xi^3||).$$

Combining equation (13) and (20), we obtain the second-order approximation of the welfare objective, which is given by

$$W_0 = E_0\sum_{t=0}^{\infty}\beta^t u_c(\bar{C})\bar{C}\{\hat{c}_t + \frac{1}{2}(1-\sigma)\hat{c}_t{}^2 + A_1\hat{y}_t + A_2\hat{y}_t^2 + A_3\hat{\pi}_{H,t}^2 + A_4\hat{a}_t\hat{y}_t + f.t\} + t.i.p + \mathcal{O}(||\xi^3||), \tag{21}$$

where

$$A_1 = -\frac{\epsilon-1}{\epsilon},$$
$$A_2 = -\frac{\epsilon-1}{\epsilon}\frac{1}{2}(1+\psi)(\frac{1}{\alpha_p}),$$

$$A_3 = -(\epsilon - 1)\epsilon\frac{1}{2}[-(1 - \frac{1}{\epsilon}) + (1+\psi)(\frac{1}{\alpha_p})]\frac{\theta}{(1-\theta)(1-\beta\theta)},$$

$$A_4 = \frac{\epsilon - 1}{\epsilon}(1+\psi)(\frac{1}{\alpha_p}).$$

To solve for the optimality, we then need to derive $f.t.$ Specifically, we have to consider the derivatives $\partial\hat{s}_t\partial\hat{y}_{t+i}$, $\partial\hat{p}_{H,t}\partial\hat{y}_{t+i}$, $\partial\hat{y}_t\partial\hat{y}_{t+i}$, $\partial\hat{s}_t\partial\hat{a}_{t+i}$, $\partial\hat{p}_{H,t}\partial\hat{a}_{t+i}$, $\partial\hat{y}_t\partial\hat{a}_{t+i}$, for all i from one to infinity. All these terms occur because the level of R&D is determined by output, the terms of trade and technology. Note that consumption will not directly affect the R&D level, and therefore does not generate joint derivatives. For the second-order approximation, given terms that are already partial derivatives of current variables, I use equations that hold only up to first-order approximation when calculating the joint derivatives since inaccuracy of those terms will become of third-order. Therefore, I assume $\hat{y}_t = E_t(\hat{y}_t(i))$ throughout the derivation of $f.t.$

Derivation of $f.t$

First, note that up to the first order-approximation, the optimality conditions of firms and the condition of the complete market can be given by

$$\hat{y}_t = \hat{a}_t + \alpha_p\hat{n}_t, \tag{22}$$

$$\hat{y}_t = \hat{c}_t + \frac{\alpha\omega}{\sigma}\hat{s}_t, \tag{23}$$

$$\hat{y}_t = \hat{y}_t^* + \frac{1}{\sigma_\alpha}\hat{s}_t. \tag{24}$$

On the other hand, the optimality condition of household is given

$$\begin{aligned}
Y_t &= C_t S_t^{(\alpha\omega/\sigma)} \\
&= C_t^{1+\frac{\alpha\omega}{1-\alpha}} Y_t^{*\frac{-\alpha\omega}{1-\alpha}}, \\
\Rightarrow C_t &= Y(N_t, A_t)^{1/(1+\frac{\alpha\omega}{1-\alpha})} Y_t^{*(\frac{-\alpha\omega}{1-\alpha})/(1+\frac{\alpha\omega}{1-\alpha})}, \\
\Rightarrow C_t^\sigma N_t^\psi &= -\frac{U_N}{U_C} = \frac{\alpha_p}{(1+\frac{\alpha\omega}{1-\alpha})}\frac{C_t}{N_t}, \\
\Rightarrow (1-\sigma)\hat{c}_t &= (1+\psi)\hat{n}_t. \qquad (25)
\end{aligned}$$

Using equation (25), (22), (23) and (24), we can solve for $\hat{y}_t$, $\hat{c}_t$, $\hat{n}_t$ and $\hat{s}_t$ as functions of $\hat{a}_t$ and $\hat{y}^*_t$. Next, I consider the expected gap of aggregate variables at period $t+1$. The expected gap of output is given by

$$\begin{aligned} E_t\hat{y}_{t+1} &= E_t\Omega_a\hat{a}_{t+1} - \Omega^*_y E_t\hat{y}^*_{t+1} \\ &= \Omega_a(\alpha_1\hat{a}_t + \alpha_2\hat{d}_t) - \Omega^*_y\rho_y y^*_t, \end{aligned}$$

where

$$\begin{aligned} \Omega^*_y &= -(1 - \frac{\sigma}{\sigma_\alpha}\frac{1}{\alpha\omega}(1 - \frac{1+\psi}{1-\sigma}\frac{1}{\alpha_p}))^{-1}, \\ \Omega_a &= -\Omega^*_y(\frac{\sigma}{\sigma_\alpha}\frac{1}{\alpha\omega}(1 - \frac{1+\psi}{1-\sigma}\frac{1}{\alpha_p})). \end{aligned}$$

The expected gap of the terms of trade is given by

$$\begin{aligned} E_t\hat{s}_{t+1} &= E_t[\sigma_\alpha(\hat{y}_{t+1} - \hat{y}^*_{t+1})] \\ &= \sigma_\alpha(\Omega_a(\alpha_1\hat{a}_t + \alpha_2\hat{d}_t) - \Omega_{y^*}\rho_y\hat{y}^*_t) - \sigma_\alpha\rho_y\hat{y}^*_t. \end{aligned} \tag{26}$$

From the Phillips curve and the natural rate of output (9), expected domestic inflation is given by

$$E_t\pi_{H,t+1} = \frac{1}{\beta}(\pi_{H,t} - (\hat{y}_t - \hat{y}^n_t)) = \frac{1}{\beta}\pi_{H,t} - \frac{1}{\beta}\hat{y}_t + \frac{1}{\beta}\Gamma_a\hat{a}_t + \frac{1}{\beta}\Gamma_*\hat{y}^*_t. \tag{27}$$

Following the functional form of R&D derived by Chi (2018), the R&D gap is given by

$$\hat{d}_t = \frac{1}{den}[(c_4 + c_7)E_t\pi_{H,t+1} + c_4\alpha E_t\hat{s}_{t+1} + \alpha\hat{s}_t + c_6\rho_y\hat{y}^*_t + c_9\hat{a}_t, \tag{28}$$

where $den = ln\epsilon_d + 1 + \alpha_2(1+\psi)/(1-\alpha_p) - exp(\alpha_2/(1-\alpha_p))$. Plugging equation (28) into equation (26) and combining equation (27), we can then obtain the matrix that describes the evolution of gaps of aggregate variables:

$$E_t \begin{bmatrix} \hat{y}_{t+1} \\ \pi_{H,t+1} \\ \hat{s}_{t+1} \\ \hat{a}_{t+1} \\ \hat{y}^*_{t+1} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} \\ M_{41} & M_{42} & M_{43} & M_{44} & M_{45} \\ M_{51} & M_{52} & M_{53} & M_{54} & M_{55} \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \pi_{H,t} \\ \hat{s}_t \\ \hat{a}_t \\ \hat{y}^*_t \end{bmatrix} \triangleq M b_t, \tag{29}$$

where

$$M_{31} = \frac{\Omega_a \sigma_\alpha \alpha_2}{den}(\alpha\sigma_\alpha - (c_4 + c_7)\frac{K}{\beta})\frac{1}{F_2},$$

$$M_{32} = \frac{\Omega_a \sigma_\alpha \alpha_2}{den}\frac{(c_4 + c_7)}{\beta}\frac{1}{F_2},$$

$$M_{33} = 0,$$

$$M_{34} = [\sigma_\alpha \Omega_a \alpha_1 + \frac{\sigma_\alpha \alpha_2 \Omega_a}{den} c_9 + \frac{\sigma_\alpha \alpha_2 \Omega_a}{den}(c_4 + c_7)\frac{\Gamma_a}{\beta}]\frac{1}{F_2},$$

$$M_{35} = [\frac{\sigma_\alpha \alpha_2 \Omega_a}{den}(c_6 \rho_y - \alpha\sigma_\alpha + \frac{c_4 + c_7}{\beta}\Gamma_*) - \sigma_\alpha \rho_y (1 + \Omega_{y^*})]\frac{1}{F_2},$$

$$M_{11} = \frac{1}{\sigma_\alpha} M_{31}, \quad M_{12} = \frac{1}{\sigma_\alpha} M_{32}, \quad M_{13} = 0, \quad M_{14} = \frac{1}{\sigma_\alpha} M_{34}, \quad M_{15} = (1 + \frac{1}{\sigma_\alpha} M_{35}),$$

$$M_{21} = \frac{K}{\beta}, \quad M_{22} = \frac{1}{\beta}, \quad M_{23} = 0, \quad M_{24} = \frac{K}{\beta}\Gamma_a, \quad M_{25} = \frac{K}{\beta}\Gamma_*,$$

$$M_{41} = \frac{\alpha_2 (c_4 + c_7)}{den} M_{21} + \frac{\alpha_2 \alpha c_4}{den} M_{31},$$

$$M_{42} = \frac{\alpha_2 (c_4 + c_7)}{den} M_{22} + \frac{\alpha_2 \alpha c_4}{den} M_{32},$$

$$M_{43} = \frac{\alpha_2}{den}\alpha,$$

$$M_{44} = \alpha_1 + \frac{\alpha_2 (c_4 + c_7)}{den} M_{24} + \frac{\alpha_2 \alpha c_4}{den} M_{34} + \frac{\alpha_2}{den} c_9,$$

$$M_{45} = \frac{\alpha_2 (c_4 + c_7)}{den} M_{25} + \frac{\alpha_2 \alpha c_4}{den} M_{35} + \frac{\alpha_2}{den} c_6 \rho_y,$$

$$M_{51} = M_{52} = M_{53} = M_{54} = 0, \quad M_{55} = \rho_y,$$

$$F_2 = (1 - \frac{\sigma_\alpha \alpha_2 \Omega_a}{den} c_4 \alpha).$$

To see how the current gaps of aggregate variables correlates with future gaps via R&D,

I will first examine the evolution of endogenous technology. Using equation (28), the technology process is given by

$$E_t a_{t+i} = \alpha_0 + \alpha_1 E_t a_{t+i-1} + \frac{\alpha_2}{den}[c_3 + (c_4+c_7)E_t\pi_{H,t+i} + c_4\alpha E_t\hat{s}_{t+i} + \alpha E_t\hat{s}_{t+i-1}$$

$$+ c_6\rho_y\hat{y}^*_{t+i-1} + c_9\hat{a}_{t+i-1} + (c_4\alpha+\alpha)\bar{s} + \bar{a}c_9]$$

$$= \alpha_1 M^{i-1}_{[4,;]}b_t + \frac{\alpha_2}{den}[(c_4+c_7)M^i_{[2,;]}b_t + c_4\alpha M^i_{[3,;]}b_t + \alpha M^{i-1}_{[3,;]}b_t + c_6\rho_y M^{i-1}_{[5,;]}b_t + c_9 M^{i-1}_{[4,;]}b_t] + constant,$$

$$M^i_a \triangleq \alpha_1 M^{i-1}_{[4,;]}b_t + \frac{\alpha_2}{den}[(c_4+c_7)M^i_{[2,;]}b_t + c_4\alpha M^i_{[3,;]}b_t + \alpha M^{i-1}_{[3,;]}b_t + c_6\rho_y M^{i-1}_{[5,;]}b_t + c_9 M^{i-1}_{[4,;]}b_t].$$

where

$$b_t = \begin{bmatrix} \hat{y}_t \\ \pi_{H,t} \\ \hat{s}_t \\ \hat{a}_t \\ \hat{y}^*_t \end{bmatrix}, \quad M^{i-1}_{[x,;]} = p_x M^{i-1},$$

and p_x indicates a zero row in dimension 1×5, with $[1,x]$ being 1 (e.g. $p_1 = [1,0,0,0,0]$). The derivative with respect to current s_t and future y_{t+i} can thus be given by

$$[\hat{s}_t\hat{y}_{t+1}]: \quad \beta[\psi\bar{N}^{\psi-1}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{2/\alpha_p} + \bar{N}^{\psi}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}]\frac{\alpha_2\alpha}{den}\hat{s}_t\hat{y}_{t+1},$$

$$[\hat{s}_t\hat{y}_{t+i}]: \quad \beta^i[\psi\bar{N}^{\psi-1}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{2/\alpha_p} + \bar{N}^{\psi}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}]M^i_{a[1,3]}\hat{s}_t\hat{y}_{t+1} \quad if \quad i>1.$$

According to the matrix, we know that $\hat{y}_{t+i} = M^i_{[1.;]}b_t$, and therefore by summing all the joint partial derivatives with respect to $\hat{s}_t$ and $\hat{y}_{t+i}$ we obtain

$$\hat{s}_t\hat{y}_{t+1} + ... + \hat{s}_t\hat{y}_\infty = [\psi\bar{N}^{\psi-1}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{2/\alpha_p} + \bar{N}^{\psi}(\frac{1}{\alpha_p})^2(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}][\frac{\beta\alpha_2\alpha}{den}M_{[1,;]}b_t\hat{s}_t + \sum_{i=2}^{\infty} M^i_{a[1,3]}\beta^i\hat{s}_t M^i_{[1,;]}b_t] \tag{30}$$

$$= \bar{N}^{\psi}\frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}(\frac{1+\psi}{\alpha_p})[\frac{\beta\alpha_2\alpha}{den}M_{[1,;]}b_t\hat{s}_t + \sum_{i=2}^{\infty} M^i_{a[1,3]}\beta^i\hat{s}_t M^i_{[1,;]}b_t]. \tag{31}$$

Similarly,

$$\hat{p}_{H,t}\hat{y}_{t+1} + ... + \hat{p}_{H,t}\hat{y}_{\infty} = \bar{N}^{\psi}\frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}(\frac{1+\psi}{\alpha_p})[c_p M_{[1,;]}b_t\hat{p}_{H,t} + \sum_{i=2}^{\infty} M^i_{a[1,2]}\beta^i\hat{p}_{H,t}M^i_{[1,;]}b_t] \tag{32}$$

$$\hat{y}_t\hat{y}_{t+1} + ... + \hat{y}_t\hat{y}_{\infty} = \bar{N}^{\psi}\frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}(\frac{1+\psi}{\alpha_p})[\frac{\beta\alpha_2}{den}(c_4+c_7)M_{21}M_{[1.;]}b_t\hat{y}_t + \sum_{i=2}^{\infty} M^i_{a[1,1]}\beta^i\hat{y}_t M^i_{[1,;]}b_t \tag{33}$$

$$\hat{s}_t\hat{a}_{t+1} + ... + \hat{s}_t\hat{a}_{\infty} = \bar{N}^{\psi}\frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}(\frac{1+\psi}{\alpha_p})[\frac{\beta\alpha_2\alpha}{den}M_{[4,;]}b_t\hat{s}_t + \sum_{i=2}^{\infty} M^i_{a[1,3]}\beta^i\hat{s}_t M^i_{[4,;]}b_t] \tag{34}$$

$$\hat{p}_{H,t}\hat{a}_{t+1} + ... + \hat{p}_{H,t}\hat{a}_{\infty} = \bar{N}^{\psi}\frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}(\frac{1+\psi}{\alpha_p})[c_p M_{[4,;]}b_t\hat{p}_{H,t} + \sum_{i=2}^{\infty} M^i_{a[1,2]}\beta^i\hat{p}_{H,t}M^i_{[4,;]}b_t] \tag{35}$$

$$\hat{y}_t\hat{a}_{t+1} + ... + \hat{y}_t\hat{a}_{\infty} = \bar{N}^{\psi}\frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}(\frac{1+\psi}{\alpha_p})[\frac{\beta\alpha_2}{den}(c_4+c_7)M_{21}M_{[4.;]}b_t\hat{y}_t + \sum_{i=2}^{\infty} M^i_{a[1,1]}\beta^i\hat{y}_t M^i_{[4,;]}b_t \tag{36}$$

$$where \;\; c_p = \frac{\beta[(c_4+c_7)M_{22} + c_4\alpha M_{32}]\alpha_2}{den} \tag{37}$$

Next, we calculate the geometric sum of equation (30) to (37). Note that M^i_a (which is a 1×5 matrix) can be written as

$$\begin{aligned} M^i_a &= \{\alpha_1 p_4 + \frac{\alpha_2}{den}[(c_4+c_7)p_2 M + c_4\alpha p_3 M + \alpha p_3 + c_6\rho_y p_5 + c_9 p_4]\} \times M^i \\ &\triangleq f_a(M) \times M^i, \end{aligned}$$

and therefore equation (30) can be rearranged as

$$\hat{s}_t\hat{y}_{t+1} + ... + \hat{s}_t\hat{y}_{\infty} = \bar{N}^{\psi}\frac{1}{\alpha_p}(\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p}[\frac{\beta\alpha_2\alpha}{den}p_1 M b_t\hat{s}_t + \sum_{i=1}^{\infty} f_a(M)M^i p_3^{'}\beta^i p_1 M^i M b_t\hat{s}_t]. \tag{38}$$

By assuming $T^{31} = p_3^{'}p_1$,

$$\sum_{i=1}^{\infty} M^i p_3^{'} \beta^i p_1 M^i \triangleq Z^{31},$$

$$Z^{31} \triangleq (\beta^{1/2}M)^1 T^{31} (\beta^{1/2}M)^1 + (\beta^{1/2}M)^2 T^{31} (\beta^{1/2}M)^2 + ...,$$

$$(\beta^{1/2}M) Z^{31} (\beta^{1/2}M) = (\beta^{1/2}M)^2 T^{31} (\beta^{1/2}M)^2 + (\beta^{1/2}M)^3 T^{31} (\beta^{1/2}M)^3 + ...,$$

$$\Rightarrow Z^{31} - (\beta^{1/2}M) Z^{31} (\beta^{1/2}M) = (\beta^{1/2}M) T^{31} (\beta^{1/2}M). \tag{39}$$

The value of Z^{31} can be solved by equation (39). Plugging Z^{31} into equation (38) we obtain

$$\hat{s}_t \hat{y}_{t+1} + ... + \hat{s}_t \hat{y}_{\infty} = \bar{N}^{\psi} \frac{1}{\alpha_p} (\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p} (\frac{1+\psi}{\alpha_p}) [\frac{\beta \alpha_2 \alpha}{den} p_1 M + f_a(M) Z^{31} M] b_t \hat{s}_t \tag{40}$$

Similarly,

$$\hat{p}_{H,t} \hat{y}_{t+1} + ... + \hat{p}_{H,t} \hat{y}_{\infty} = \bar{N}^{\psi} \frac{1}{\alpha_p} (\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p} (\frac{1+\psi}{\alpha_p}) [c_p p_1 M + f_a(M) Z^{21} M] \hat{p}_{H,t} b_t \tag{41}$$

$$\hat{y}_t \hat{y}_{t+1} + ... + \hat{y}_t \hat{y}_{\infty} = \bar{N}^{\psi} \frac{1}{\alpha_p} (\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p} (\frac{1+\psi}{\alpha_p}) [\frac{\beta \alpha_2}{den} (c_4 + c_7) M_{21} p_1 M + f_a(M) Z^{11} M] \hat{y}_t b_t \tag{42}$$

$$\hat{s}_t \hat{a}_{t+1} + ... + \hat{s}_t \hat{a}_{\infty} = -\bar{N}^{\psi} \frac{1}{\alpha_p} (\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p} (\frac{1+\psi}{\alpha_p}) [\frac{\beta \alpha_2 \alpha}{den} p_4 M + f_a(M) Z^{34} M] b_t \hat{s}_t \tag{43}$$

$$\hat{p}_{H,t} \hat{a}_{t+1} + ... + \hat{p}_{H,t} \hat{a}_{\infty} = -\bar{N}^{\psi} \frac{1}{\alpha_p} (\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p} (\frac{1+\psi}{\alpha_p}) [c_p p_4 M + f_a(M) Z^{24} M] b_t \hat{p}_{H,t} \tag{44}$$

$$\hat{y}_t \hat{a}_{t+1} + ... + \hat{y}_t \hat{a}_{\infty} = -\bar{N}^{\psi} \frac{1}{\alpha_p} (\frac{\bar{Y}}{\bar{A}})^{1/\alpha_p} (\frac{1+\psi}{\alpha_p}) [\frac{\beta \alpha_2}{den} (c_4 + c_7) M_{21} p_4 M + f_a(M) Z^{14} M] b_t \hat{y}_t \tag{45}$$

Finally, by summing up (40) to (45) we get $f.t$.

With the function form of $f.t$ in hand, we can continue the second order approximation of the welfare function. Let $g_t = \begin{bmatrix} \hat{y}_t & \hat{c}_t & \hat{p}_{H,t} & \hat{s}_t \end{bmatrix}'$ be the vector of aggregate variables and $e_t = \begin{bmatrix} \hat{a}_t & \hat{y}_t^* \end{bmatrix}'$ be the set of shocks, the welfare objective can finally be given by

$$W_0 = E_0 \sum_{t=0}^{\infty} \beta^t u_c(\bar{C})\bar{C}[w_g^1 g_t - \frac{1}{2} g_t^{'} w_g^2 g_t - g_t^{'} w_e e_t - \frac{1}{2} w_\pi \hat{\pi}_{H,t}^2 - g_t^{\prime} w_\pi^0 \hat{\pi}_{H,t}],$$

where

$$w_g^1 = \begin{bmatrix} -\frac{\epsilon-1}{\epsilon} & 1 & 0 & 0 \end{bmatrix},$$

$$w_g^2 = \begin{bmatrix} \frac{1-\epsilon}{\epsilon}(1+\psi)(\frac{1}{\alpha_p}) + M_{[1,1]}^{by} & 0 & M_{[1,1]}^{bp} & M_{[1,1]}^{bs} + M_{[1,3]}^{by} \\ 0 & -(1-\sigma) & 0 & 0 \\ M_{[1,1]}^{bp} & 0 & 0 & M_{[1,3]}^{bp} \\ M_{[1,1]}^{bs} + M_{[1,3]}^{by} & 0 & M_{[1,3]}^{bp} & M_{[1,3]}^{bs} \end{bmatrix},$$

$$w_e = \begin{bmatrix} -\frac{\epsilon-1}{\epsilon}(1+\psi)(\frac{1}{\alpha_p}) + M_{[1,4]}^{by} & M_{[1,5]}^{by} \\ 0 & 0 \\ M_{[1,4]}^{bp} & M_{[1,5]}^{bp} \\ M_{[1,4]}^{bs} & M_{[1,5]}^{bs} \end{bmatrix},$$

$$w_\pi = (\epsilon - 1)\epsilon[-(1 - \frac{1}{\epsilon}) + (1+\psi)(\frac{1}{\alpha_p})] \frac{\theta}{(1-\theta)(1-\beta\theta)},$$

$$w_\pi^0 = \begin{bmatrix} -M_{[1,2]}^{by} & 0 & -M_{[1,2]}^{bp} & -M_{[1,2]}^{bs} \end{bmatrix}',$$

$$M^{bs} = (\frac{1+\psi}{\alpha_p}) \frac{\beta\alpha_2\alpha}{den}(p_1 - p_4)M + f_a(M)(Z^{31} - Z^{34})M,$$

$$M^{bp} = (\frac{1+\psi}{\alpha_p}) c_p (p_1 - p_4)M + f_a(M)(Z^{21} - Z^{24})M,$$

$$M^{by} = (\frac{1+\psi}{\alpha_p}) \frac{\beta\alpha_2}{den}(c_4 + c_7)M_{21}(p_1 - p_4)M + f_a(M)(Z^{11} - Z^{14})M.$$

Note that the welfare objective is affected by the gap of the terms of trade, consistent with the literature that emphasizes the terms of trade externality. Next, I will approximate the conditions that jointly solve the model.

5.1.2 Risk Sharing

By assuming that market is complete, domestic consumption is proportional to world consumption:

$$
\begin{aligned}
c_t &= c_t^* + (\frac{1-\alpha}{\sigma})s_t \\
&= y_t^* + (\frac{1-\alpha}{\sigma})s_t, \\
\Rightarrow \hat{c}_t &= \hat{y}_t^* + (\frac{1-\alpha}{\sigma})\hat{s}_t.
\end{aligned} \tag{46}
$$

The linear-quadratic form of equation (46) is given by

$$
E_0 \sum_{t=0}^{\infty} \beta^t [c_g^1 g_t + \frac{1}{2} g_t^{'} c_g^2 g_t + g_t^{'} c_e e_t] + t.i.p = 0,
$$

where

$$
\begin{aligned}
c_g^1 &= [0, -1, 0, \frac{1-\alpha}{\sigma}], \\
c_g^2 &= 0_{4\times 4}, \\
c_e &= 0_{4\times 2}.
\end{aligned}
$$

5.1.3 Exchange Rate

With the substitutability between domestic and foreign goods η, the CPI is given by

$$
P_t = [(1-\alpha)P_{H,t}^{1-\eta} + \alpha P_{F,t}^{1-\eta}]^{1/(1-\eta)}.
$$

The second-order Taylor approximation around the steady state is thus given by

$$
\begin{aligned}
0 &= \bar{P_H}\hat{p}_{H,t} + \bar{P_H}\frac{1}{1-\eta}\alpha(1-\eta)(\bar{S})^{(1-\eta)}\hat{s}_t + \bar{P_H}[\frac{1}{1-\eta}\alpha(1-\eta)^2(\bar{S})^{(1-\eta)} \\
&\quad + \frac{1}{1-\eta}(\frac{1}{1-\eta} - 1)\alpha^2(1-\eta)^2(\bar{S})^{(1-\eta)}]\hat{s}_t^2 + \bar{P_H}\alpha(\bar{S})^{(1-\eta)}\hat{s}_t\hat{p}_{H,t} \\
&= \bar{P_H}[\hat{p}_{H,t} + \alpha\hat{s}_t + \frac{1}{2}\alpha(1-\eta+\alpha\eta)\hat{s}_t^2 + \alpha\hat{s}_t\hat{p}_{H,t}].
\end{aligned} \tag{47}
$$

Equation (47) can also be written as

$$0 = E_0 \sum_{t=0}^{\infty} \beta^t [f_g^1 g_t + \frac{1}{2} g_t^{'} f_g^2 g_t + g_t^{'} f_e e_t] + t.i.p + \mathcal{O}(||\xi^3||),$$

where

$$f_g^1 = \begin{bmatrix} 0 & 0 & 1 & \alpha \end{bmatrix},$$

$$f_g^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & \alpha(1-\eta+\alpha\eta) \end{bmatrix},$$

$$f_e = 0_{4\times 2}.$$

5.1.4 Price setting

Firms choose prices to maximize the expected sum of profit:

$$max_{P_{H,t}(j)} \sum_{k=0}^{\infty} \theta^k E_t Q_{t,t+k} \{(P_{H,t}(j) Y_{t+k|t}(j) - \Phi_{t+k}(Y_{t+k|t}(j))\},$$

where Φ_{t+k} is the nominal cost in period $t+k$ and

$$Q_{t,t+k} = \beta^k (\frac{C_{t+k}}{C_t})^{-\sigma} \frac{P_t}{P_{t+k}},$$

$$Y_{t+k|t}(j) = (\frac{P_{H,t}(j)}{P_{H,t+k}})^{-\epsilon} Y_{t+k},$$

where $Q_{t,t+k}$ is the stochastic discount factor and $Y_{t+k|t}(j)$ is production in period $t+k$ if firm j last set the price in period t. The first-order condition is

$$\sum_{k=0}^{\infty} \theta^k E_t Q_{t,t+k} [(1-\epsilon)(\frac{P_{H,t}(j)}{P_{H,t+k}})^{-\epsilon} Y_{t+k} - \phi(Y_{t+k|t}(j))(-\epsilon) \frac{P_{H,t}(j)^{-\epsilon-1}}{P_{H,t+k}^{-\epsilon}} Y_{t+k}] = 0, \quad (48)$$

$$\Rightarrow \sum_{k=0}^{\infty} \theta^k E_t Q_{t,t+k} [(1-\epsilon)(\frac{1}{P_{H,t+k}})^{-\epsilon} Y_{t+k} - \phi(Y_{t+k|t}(j))(-\epsilon) \frac{P_{H,t}(j)^{-1}}{P_{H,t+k}^{-\epsilon}} Y_{t+k}] = 0, \quad (49)$$

where $\phi = ln(\Phi)$. Let ϕ^R denote the log of the real marginal cost, the functional form of marginal costs are given by

$$\phi(Y_{t+k|t}(j)) = \frac{(1-\tau_{t+k})W_{t+k}}{\alpha_p N_{t+k|t}^{\alpha_p - 1} A_{t+k}(j)}$$
$$\phi^R(Y_{t+k|t}(j)) = \frac{1}{P_{t+k}} \frac{(1-\tau_{t+k})W_{t+k}}{\alpha_p N_{t+k|t}^{\alpha_p - 1} A_{t+k}(j)} \tag{50}$$

Note that the associated labor demand are given by

$$\begin{aligned} N_{t+k|t}(j) &= (Y_{t+k|t}(j))^{1/\alpha_p}(A_{t+k}(j))^{-1/\alpha_p} \\ &= (Y_{t+k})^{1/\alpha_p}(\frac{P_{H,t}(j)}{P_{H,t+k}})^{-\epsilon/\alpha_p}(A_{t+k}(j))^{-1/\alpha_p}, \end{aligned} \tag{51}$$

$$\begin{aligned} N_{t+k} &= (\frac{Y_{t+k}}{A_{t+k}})^{1/\alpha_p} \int_0^1 (\frac{P_{H,t+k}(j)}{P_{H,t+k}})^{-\epsilon/\alpha_p} dj \\ &\triangleq (\frac{Y_{t+k}}{A_{t+k}})^{1/\alpha_p} D_{t+k}. \end{aligned} \tag{52}$$

The evolution of the domestic price is given by

$$P_{H,t}^{1-\epsilon} = \theta P_{H,t-1}^{1-\epsilon} + (1-\theta)P_{H,t}(j)^{1-\epsilon}. \tag{53}$$

The optimality condition of the household is given by

$$\frac{W_{t+k}}{P_{t+k}} = \frac{-U_N}{U_C} = \frac{N_{t+k}^{\psi}}{C_{t+k}^{-\sigma}} \tag{54}$$

By plugging equation (51), (52), (53), and (54), the real marginal cost (50) can further be given by

$$\phi^R(Y_{t+k|t}(j)) = \frac{1-\tau_{t+k}}{\alpha_p} Y_{t+k}^{\frac{\psi+1-\alpha_p}{\alpha_p}} A_{t+k}^{-(1+\frac{\psi+1-\alpha_p}{\alpha_p})} C_{t+k}^{\sigma} (\frac{P_{H,t}(j)}{P_{H,t+k}})^{\frac{-\epsilon(1-\alpha_p)}{\alpha_p}} D_{t+k}^{\psi}. \tag{55}$$

Plugging the real marginal cost (55) into the first-order condition (48) we obtain

$$\sum_{k=0}^{\infty}(\theta\beta)^k E_t Q_{t,t+k}[\frac{1}{P_{H,t+k}}]^{-\epsilon} Y_{t+k}$$
$$\times[\frac{1}{P_{t+k}} - \frac{\epsilon(1-\tau_{t+k})}{\epsilon-1}\frac{1}{\alpha_p} Y_{t+k}^{\frac{\psi+1-\alpha_p}{\alpha_p}} A_{t+k}^{-(1+\frac{\psi+1-\alpha_p}{\alpha_p})} C_{t+k}^{\sigma} P_{H,t}(j)^{\frac{-\epsilon(1-\alpha_p)}{\alpha_p}-1}(\frac{1}{P_{H,t+k}})^{\frac{-\epsilon(1-\alpha_p)}{\alpha_p}} D_{t+k}^{\psi}] = 0 \tag{56}$$

$$\Rightarrow \mathcal{L}_t \triangleq \sum_{k=0}^{\infty}(\theta\beta)^k E_t Q_{t,t+k} P_{H,t+k}^{\epsilon} Y_{t+k} \tag{57}$$
$$\times[\frac{1}{P_{t+k}} - \frac{\epsilon(1-\tau_{t+k})}{\epsilon-1}\frac{1}{\alpha_p} Y_{t+k}^{\frac{\psi+1-\alpha_p}{\alpha_p}} A_{t+k}^{-(1+\frac{\psi+1-\alpha_p}{\alpha_p})} C_{t+k}^{\sigma}(\frac{P_{H,t}^{1-\epsilon} - \theta P_{H,t-1}^{1-\epsilon}}{1-\epsilon})^{\frac{-\epsilon(1-\alpha_p)}{(1-\epsilon)\alpha_p}} P_{H,t+k}{}^{\frac{\epsilon(1-\alpha_p)}{\alpha_p}} D_{t+k}^{\psi}] = 0, \tag{58}$$

where $\mathcal{L}_t$ denotes the price setting equation. According to (53), the term D_{t+k} is given by

$$D_{t+k}^{\psi} = [\frac{(\frac{P_{H,t+k}^{1-\epsilon} - \theta P_{H,t+k-1}^{1-\epsilon}}{1-\theta})^{1/(1-\epsilon)}}{P_{H,t+k}}]^{\frac{\psi\epsilon}{1-\alpha_p}}$$
$$= [\frac{(1-\theta(P_{H,t+k-1}/P_{H,t+k})^{1-\epsilon})^{1/(1-\epsilon)}}{1-\theta}]^{\frac{\psi\epsilon}{1-\alpha_p}}.$$

For simplicity, I will use the fact that $E_t(\theta\beta\mathcal{L}_{t+1}) - \mathcal{L}_t = 0$ and obtain

$$P_{H,t}^{\epsilon} Y_t P_t^{-1} = E_t \sum_{k=0}^{\infty}[(\theta\beta)^k Q_{t,t+k}(G_{t+k}\eta_{t,t-1} - \theta\beta G_{t+k+1}\eta_{t+1,t})], \tag{59}$$

where

$$G_{t+k} = [\frac{\epsilon(1-\tau_{t+k})}{\alpha_p(1-\epsilon)}] P_{H,t+k}^{\epsilon/\alpha_p - \psi\epsilon/(1-\alpha_p)} Y_{t+k}^{(1+\psi)/\alpha_p} A_{t+k}^{-(1+\psi)/\alpha_p} \eta_{t+k,t+k-1}^{-\psi\alpha_p/(1-\alpha_p)^2} P_{H,t+k}^{-\psi\epsilon/(1-\alpha_p)} c_{t+k}^{\sigma},$$
$$\eta_{t+k,t+k-1} = \frac{P_{H,t+k}^{1-\epsilon} - \theta P_{H,t+k-1}^{1-\epsilon}}{1-\theta},$$
$$\phi = \frac{-\epsilon(1-\alpha_p)}{(1-\epsilon)\alpha_p}.$$

The right hand side of equation (59) can be written as

$$G_t\eta_{t,t-1} - \theta\beta E(G_{t+1}\eta_{t+1,t}) + E_t\sum_{k=1}^{\infty}[(\theta\beta)^k Q_{t,t+k}(G_{t+k}\eta_{t,t-1} - \theta\beta G_{t+k+1}\eta_{t+1,t})],$$

$$= G_t\eta_{t,t-1} - \theta\beta E(G_{t+1}\eta_{t+1,t}) + \theta\beta E_t Q_{t,t+1}E_{t+1}\sum_{k=0}^{\infty}[(\theta\beta)^k Q_{t+1,t+1+k}(G_{t+1+k}\eta_{t+1,t} - \theta\beta G_{t+1+k+1}\eta_{t+2,t+1})],$$

$$= G_t\eta_{t,t-1} - \theta\beta E(G_{t+1}\eta_{t+1,t}) + \theta\beta E_t Q_{t,t+1}P^{\epsilon}_{H,t+1}Y_{t+1}P^{-1}_{t+1}.$$

The first order condition can then be given by

$$P^{\epsilon}_{H,t}Y_tP^{-1}_t = G_t\eta_{t,t-1} - \theta\beta E(G_{t+1}\eta_{t+1,t}) + \theta\beta E_t Q_{t,t+1}P^{\epsilon}_{H,t+1}Y_{t+1}P^{-1}_{t+1}. \tag{60}$$

Around symmetric steady state where $\bar{S} = 1$, the linear-quadratic form of the left-hand side of equation (60) is given by

$$P^{\epsilon}_{H,t}Y_tP^{-1}_t = \bar{P}^{\epsilon-1}_H\bar{Y} \times [o^1_g g_t + \frac{1}{2}g^{'}_t o^2_g g_t]$$

where

$$o^1_g = \begin{bmatrix} 1 & 0 & (\epsilon-1) & -\alpha \end{bmatrix},$$

$$o^2_g = \begin{bmatrix} 1 & 0 & (\epsilon-1) & -\alpha \\ 0 & 0 & 0 & 0 \\ (\epsilon-1) & 0 & (\epsilon-1)^2 & -(\epsilon-1)\alpha \\ -\alpha & 0 & -(\epsilon-1)\alpha & [(2-\eta)\alpha^2 + \alpha(1-\eta)] \end{bmatrix},$$

Regarding the right-hand side of equation (60), I approximate each term separately:

$$G_t\eta_{t,t-1} = [\frac{\epsilon(1-\tau_t)}{\alpha_p(1-\epsilon)}]\bar{C}^{\xi_c}\bar{P}_H^{\xi_p+(1-\epsilon)\phi\xi_\eta}\bar{Y}^{\xi_y}$$
$$\times [j^1_g g_t + \frac{1}{2}g^{'}_t j^2_g g_t + g^{'}_t j_e e_t + \frac{1}{2}j_\pi\hat{\pi}^2_{H,t} + g^{'}_t j^0_\pi\hat{\pi}_{H,t} + j^1_\pi e_t\hat{\pi}_{H,t} + c_\pi\hat{\pi}_{H,t}] + t.i.p,$$

where

$$j_g^1 = \begin{bmatrix} \xi_y & \xi_c & (\xi_p + \xi_\eta \phi(1-\epsilon)) & 0 \end{bmatrix},$$

$$j_g^2 = \begin{bmatrix} \xi_y^2 & \xi_c \xi_y & \xi_y[\xi_p + \xi_\eta \phi(1-\epsilon)] & 0 \\ \xi_c \xi_y & \xi_c^2 & \xi_c[\xi_p + \xi_\eta \phi(1-\epsilon)] & 0 \\ \xi_y[\xi_p + \xi_\eta \phi(1-\epsilon)] & \xi_c[\xi_p + \xi_\eta \phi(1-\epsilon)] & [\xi_p + \xi_\eta \phi(1-\epsilon)]^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$j_e = \begin{bmatrix} \xi_y \xi_a & 0 \\ \xi_c \xi_a & 0 \\ \xi_a[\xi_p + \xi_\eta \phi(1-\epsilon)] & 0 \\ 0 & 0 \end{bmatrix},$$

$$j_\pi = -\frac{\xi_\eta \phi(1-\epsilon)}{(1-\theta)^2},$$

$$j_\pi^0 = \begin{bmatrix} \frac{\xi_y \xi_\eta \phi \theta}{(1-\theta)} & \frac{\xi_c \xi_\eta \phi \theta}{(1-\theta)} & \frac{[\xi_p + \xi_\eta \phi(1-\epsilon)] \xi_\eta \phi \theta}{(1-\theta)} & 0 \end{bmatrix}',$$

$$j_\pi^1 = \begin{bmatrix} \frac{\xi_a \xi_\eta \phi \theta}{(1-\theta)} & 0 \end{bmatrix},$$

$$c_\pi = \xi_\eta \phi \theta (1-\theta)^{-1},$$

and

$$\begin{bmatrix} \xi_c & \xi_p & \xi_y & \xi_a & \xi_\eta \end{bmatrix} = \begin{bmatrix} \sigma & \frac{\epsilon}{\alpha_p} - \frac{\psi \epsilon}{1-\alpha_p} & \frac{1+\psi}{\alpha_p} & -\frac{1+\psi}{\alpha_p} & \frac{-\alpha_p \psi}{(1-\alpha_p)^2} + 1 \end{bmatrix}.$$

Regarding $G_{t+1}\eta_{t+1,t}$, I first use the matrix (29), where

$$E_t g_{t+1} = \begin{bmatrix} M_{11} & 0 & 0 & M_{13} \\ M_{51} + \frac{1-\alpha}{\sigma} M_{31} & 0 & 0 & M_{53} + \frac{1-\alpha}{\sigma} M_{33} \\ M_{21} & 0 & 1 & M_{23} \\ M_{31} & 0 & 0 & M_{33} \end{bmatrix} g_t + \begin{bmatrix} M_{14} & M_{15} \\ M_{54} + \frac{1-\alpha}{\sigma} M_{34} & M_{55} + \frac{1-\alpha}{\sigma} M_{35} \\ M_{24} & M_{25} \\ M_{34} & M_{35} \end{bmatrix} e_t$$

$$+ \begin{bmatrix} M_{12} \\ M_{52} + \frac{1-\alpha}{\sigma} M_{32} \\ M_{22} \\ M_{32} \end{bmatrix} \pi_{H,t}$$

$$\triangleq \vartheta_g^g g_t + \vartheta_e^g e_t + \vartheta_\pi^g \pi_{H,t},$$

$$
\begin{aligned}
E_t\hat{\pi}_{H,t+1} &= \begin{bmatrix} M_{21} & 0 & 0 & M_{23} \end{bmatrix} g_t + \begin{bmatrix} M_{24} & M_{25} \end{bmatrix} e_t + M_{22}\pi_{H,t} \\
&\triangleq \vartheta_g^\pi g_t + \vartheta_e^\pi e_t + \vartheta_\pi^\pi \pi_{H,t}, \\
E_t e_{t+1} &= \begin{bmatrix} M_{41} & 0 & 0 & M_{43} \\ M_{51} & 0 & 0 & M_{53} \end{bmatrix} g_t + \begin{bmatrix} M_{44} & M_{45} \\ M_{54} & M_{55} \end{bmatrix} e_t + \begin{bmatrix} M_{42} \\ M_{52} \end{bmatrix} \\
&\triangleq \vartheta_g^e g_t + \vartheta_e^e e_t + \vartheta_\pi^e \pi_{H,t}, \\
\Rightarrow E_t \begin{bmatrix} g_{t+1} \\ \hat{\pi}_{H,t+1} \\ e_{t+1} \end{bmatrix} &= \vartheta \begin{bmatrix} g_t \\ \hat{\pi}_{H,t} \\ e_t \end{bmatrix} \quad where\ \vartheta = \begin{bmatrix} \vartheta_g^g & \vartheta_\pi^g & \vartheta_e^g \\ \vartheta_g^\pi & \vartheta_\pi^\pi & \vartheta_e^\pi \\ \vartheta_g^e & \vartheta_\pi^e & \vartheta_e^e \end{bmatrix}.
\end{aligned}
$$

Therefore $E_t G_{t+1}\eta_{t+1,t}$ can be approximated as

$$
\begin{aligned}
E_t G_{t+1}\eta_{t+1,t} = [\frac{\epsilon(1-\tau_{t+1})}{\alpha_p(1-\epsilon)}]\bar{C}^{\xi_c}\bar{P}_H^{\xi_p+(1-\epsilon)\phi\xi_\eta}\bar{Y}^{\xi_y}\bar{A}^{\xi_a}[\vartheta_g^1 g_t + \frac{1}{2}g_t^{'}\vartheta_{gg}g_t + g_t^{'}\vartheta_{ge}e_t \\
+ \frac{1}{2}\vartheta_{\pi\pi}\hat{\pi}_{H,t}^2 + g_t^{\prime}\vartheta_{g\pi}\hat{\pi}_{H,t} + e_t^{\prime}\vartheta_{e\pi}\hat{\pi}_{H,t} + \vartheta_\pi^1\hat{\pi}_{H,t}] + t.i.p,
\end{aligned}
$$

where

$$
\begin{aligned}
\vartheta_g^1 &= j_g^1\vartheta_g^g + c_\pi\vartheta_g^\pi, \\
\vartheta_\pi^1 &= j_g^1\vartheta_\pi^g + c_\pi\vartheta_\pi^\pi, \\
\vartheta_{ij} &= \vartheta_i^{g^{\prime}} j_g^2\vartheta_j^g + 2\vartheta_i^{g^{\prime}} j_e\vartheta_j^e + \vartheta_i^{\pi^{\prime}} j_\pi\vartheta_j^\pi + 2\vartheta_i^{g^{\prime}} j_\pi^0\vartheta_j^\pi + 2\vartheta_i^{e^{\prime}} j_\pi^1\vartheta_j^\pi \quad for\ i,j \in \{g,e,\pi\}.
\end{aligned}
$$

The remaining term of the right-hand side of equation (60) can be approximated as

$$
\begin{aligned}
E_t Q_{t,t+1}P_{H,t+1}^{\epsilon}Y_{t+1}P_{t+1}^{-1} &= E_t\beta\bar{P}_H^{\epsilon-1}\bar{Y}[i_g^1 g_t + \frac{1}{2}g_t^{'}i_g^2 g_t + i_g^3 g_{t+1} + \frac{1}{2}g_{t+1}^{'}i_g^4 g_{t+1} + g_t^{\prime}i_g^5 g_{t+1}] \\
&= \beta\bar{P}_H^{\epsilon-1}\bar{Y}[(i_g^1 + i_g^3\vartheta_g^g)g_t + \frac{1}{2}g_t^{\prime}(i_g^2 + i_g^4 + 2i_g^5\vartheta_g^g)g_t + g_t^{'}(\vartheta_g^{g^{\prime}} i_g^4\vartheta_e^g + i_g^5\vartheta_e^g)e_t \\
&\quad + \frac{1}{2}\vartheta_\pi^{g^{\prime}} i_g^4\vartheta_\pi^g\hat{\pi}_{H,t}^2 + g_t^{\prime}(\vartheta_g^{g^{\prime}} i_g^4\vartheta_\pi^g + i_g^5\vartheta_\pi^g)\hat{\pi}_{H,t} + e_t^{\prime}(\vartheta_e^{g^{\prime}} i_g^4\vartheta_\pi^g)\hat{\pi}_{H,t} + i_g^3\vartheta_\pi^g\hat{\pi}_{H,t}] + t.i.p,
\end{aligned}
$$

where

$$i_g^1 = \begin{bmatrix} 0 & \sigma & 1 & \alpha \end{bmatrix},$$

$$i_g^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma^2 & \sigma & \sigma\alpha \\ 0 & \sigma & 1 & \alpha \\ 0 & \sigma\alpha & \alpha & \alpha^2 \end{bmatrix},$$

$$i_g^3 = \begin{bmatrix} 1 & -\sigma & (\epsilon-2) & -2\alpha \end{bmatrix},$$

$$i_g^4 = \begin{bmatrix} 1 & -\sigma & (\epsilon-2) & -2\alpha \\ -\sigma & \sigma^2 & -\sigma(\epsilon-2) & 2\sigma\alpha \\ (\epsilon-2) & -\sigma(\epsilon-2) & (\epsilon-2)^2 & -2(\epsilon-2)\alpha \\ -2\alpha & 2\sigma\alpha & -2(\epsilon-2)\alpha & 4\alpha^2 \end{bmatrix},$$

$$i_g^5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sigma & -\sigma^2 & \sigma(\epsilon-2) & -2\sigma\alpha^2 \\ 1 & -\sigma(\epsilon-2) & (\epsilon-2) & -2(\epsilon-2)\alpha \\ \alpha & \sigma\alpha & (\epsilon-2)\alpha & -2\alpha^2 \end{bmatrix}.$$

Note that at the steady state, the following equation holds:

$$[1-(\theta\beta)\beta]\bar{P}_H^{\epsilon-1}\bar{Y} = (1-\theta\beta)[\frac{\epsilon(1-\tau_{t+k})}{\alpha_p(1-\epsilon)}]\bar{C}^{\xi_c}\bar{P}_H^{\xi_p+(1-\epsilon)\phi\xi_\eta}\bar{Y}^{\xi_y}\bar{A}^{\xi_a}$$

Finally, we can then derive the second order Taylor approximation of (60) as

$$0 = E_0\sum_{t=0}^{\infty}\beta^t[a_g^1 g_t + \frac{1}{2}g_t^{'}a_g^2 g_t + g_t^{'}a_e e_t + \frac{1}{2}a_\pi\hat{\pi}_{H,t}^2 + g_t^{\prime}a_\pi^0\hat{\pi}_{H,t} + e_t^{\prime}a_\pi^1\hat{\pi}_{H,t} + a_\pi^2\hat{\pi}_{H,t}] + t.i.p + \mathcal{O}(||\xi^3||) \tag{61}$$

where

$$a_g^1 = o_g^1 - \theta\beta^2(i_g^1 + i_g^3\vartheta_g^g) - \frac{1-\theta\beta^2}{1-\theta\beta}(j_g^1 - \theta\beta\vartheta_g^1),$$

$$a_g^2 = o_g^2 - \theta\beta^2(i_g^2 + i_g^4 + 2i_g^5\vartheta_g^g) - \frac{1-\theta\beta^2}{1-\theta\beta}(j_g^2 - \theta\beta\vartheta_{gg}),$$

$$a_e = -\theta\beta^2(\vartheta_g^{g'} i_g^4 \vartheta_e^g + i_g^5 \vartheta_e^g) - \frac{1-\theta\beta^2}{1-\theta\beta}(j_e - \theta\beta\vartheta_{ge}),$$

$$a_\pi = -\theta\beta^2(\vartheta_\pi^{g'} i_g^4 \vartheta_\pi^g) - \frac{1-\theta\beta^2}{1-\theta\beta}(j_\pi - \theta\beta\vartheta_{\pi\pi}),$$

$$a_\pi^0 = -\theta\beta^2(\vartheta_g^{g'} i_g^4 \vartheta_\pi^g + i_g^5 \vartheta_\pi^g) - \frac{1-\theta\beta^2}{1-\theta\beta}(j_\pi^0 - \theta\beta\vartheta_{g\pi}),$$

$$a_\pi^1 = -\theta\beta^2(\vartheta_e^{g'} i_g^4 \vartheta_\pi^g) - \frac{1-\theta\beta^2}{1-\theta\beta}(j_\pi^1 - \theta\beta\vartheta_{e\pi}),$$

$$a_\pi^2 = -\theta\beta^2(i_g^3 \vartheta_\pi^g) - \frac{1-\theta\beta^2}{1-\theta\beta}(c_\pi - \theta\beta\vartheta_\pi^1),$$

$$G = (\psi - \alpha_p + 1)/\alpha_p,$$

$$z = -\epsilon(1-\alpha_p)/\alpha_p.$$

5.1.5 Demand

Follow Gali and Monacelli (2005), the aggregate demand condition can be written as

$$\begin{aligned} Y_t &= (\frac{P_{H,t}(j)}{P_t})^{-\epsilon}[(1-\alpha)(\frac{P_{H,t}}{P_t})^{-\eta}C_t + \alpha\int_0^1 (\frac{P_{H,t}}{\varepsilon_{i,t}P_{F,t}^i})^{-\gamma}(\frac{P_{F,t}^i}{P_t^i})^{-\eta}C_t^i di] \\ &= (\frac{P_{H,t}}{P_t})^{-\eta}C_t[(1-\alpha) + \alpha\int_0^1 (S_t^i S_{i,t})^{\gamma-\eta} Q_{i,t}^{\eta-1/\sigma} di], \\ \Rightarrow 1 &= Y_t^{-1}(\frac{P_{H,t}}{P_t})^{-\eta}C_t[(1-\alpha) + \alpha\int_0^1 (S_t^i S_{i,t})^{\gamma-\eta} Q_{i,t}^{\eta-1/\sigma} di], \end{aligned} \tag{62}$$

where

$$S_{i,t} = \frac{P_{i,t}}{P_{H,t}},$$

$$\varepsilon_{i,t} = \frac{P_{i,t}^i(j)}{P_{i,t}},$$

$$S_t^i = \frac{P_{F,t}^i}{P_{H,t}^i},$$

$$Q_{i,t} = \frac{\varepsilon_{i,t}P_t^i}{P_t},$$

and $s_t = \int s_{i,t} di \triangleq \int log S_{i,t} di$, $\int s_t^i di = \int log S_t^i di = 0$. The second order Taylor approximation of equation (62) is given by

$$0 = (-\hat{y}_t - \frac{1}{2}\hat{y}_t^2 + \hat{c}_t + \frac{1}{2}\hat{c}_t^2 + [\alpha(\eta-\gamma)-\eta]\hat{p}_{H,t} + \frac{1}{2}[\eta^2 - \alpha\gamma(\eta-\gamma)]\hat{p}_{H,t}^2 \tag{63}$$
$$+ B_s\hat{s}_t - \gamma\alpha B\hat{s}_t\hat{p}_{H,t} + \alpha B\hat{s}_t\hat{c}_t - \alpha B\hat{s}_t\hat{y}_t + t.i.p + \mathcal{O}(||\xi^3||),$$

where $B_s = [(\gamma-\eta) + (\eta - \frac{1}{\sigma})(1-\alpha)]$. Rearranging equation (63), we obtain

$$0 = E_0 \sum_{t=0}^{\infty} \beta^t [d_g^1 g_t + \frac{1}{2} g_t^{'} d_g^2 g_t + g_t^{'} d_e e_t] + t.i.p + \mathcal{O}(||\xi^3||)$$

where

$$d_g^1 = \begin{bmatrix} -1 & 1 & [\alpha(\eta-\gamma)-\eta] & B_s \end{bmatrix},$$
$$d_g^2 = \begin{bmatrix} -1 & -1 & -[\alpha(\eta-\gamma)-\eta] & -\alpha B_s \\ -1 & 1 & [\alpha(\eta-\gamma)-\eta] & \alpha B_s \\ -[\alpha(\eta-\gamma)-\eta] & [\alpha(\eta-\gamma)-\eta] & [\eta^2 - \alpha\gamma(\eta-\gamma)] & -\gamma\alpha B_s \\ -\alpha B_s & \alpha B_s & -\gamma\alpha B_s & 0 \end{bmatrix},$$
$$d_e = 0_{4\times 2}.$$

5.1.6 Derivation of the loss function

Following De Paoli (2009a), I define $L = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \end{bmatrix}'$ such that

$$\begin{bmatrix} a_g^{1'} & d_g^{1'} & f_g^{1'} & c_g^{1'} \end{bmatrix} L = w_g^{1'},$$

where

$$L_0 = E_0 \sum_{t=0}^{\infty} \beta^t u_c(\bar{C})\bar{C}[\frac{1}{2} g_t^{'} L_g g_t + g_t^{'} L_e e_t + \frac{1}{2} L_\pi \hat{\pi}_{H,t}^2 + g_t^{'} L_\pi^0 \hat{\pi}_{H,t} + L_1(e_t^{'} a_\pi^1 + a_\pi^2)\hat{\pi}_{H,t}],$$
$$L_g = w_g^2 + L_1 a_g^2 + L_2 d_g^2 + L_3 f_g^2 + L_4 c_g^2,$$
$$L_e = w_e + L_1 a_e + L_2 d_e + L_3 f_e + L_4 c_e,$$

$$L_\pi = w_\pi + L_1 a_\pi,$$

$$L_\pi^0 = w_\pi^0 + L_1 a_\pi^0.$$

The loss function is given by

$$L_0^* = E_0 \sum_{t=0}^{\infty} \beta^t u_c(\bar{C})\bar{C}[\frac{1}{2}g_t^{'} L_g g_t + g_t^{'} L_e e_t + \frac{1}{2} L_\pi \hat{\pi}_{H,t}^2 + g_t' L_\pi^0 \hat{\pi}_{H,t} + L_1(e_t' a_\pi^1 + a_\pi^2)\hat{\pi}_{H,t}], \quad (64)$$

where the social planner aims to minimize subject to

$$\pi_{H,t} = \beta E_t \pi_{H,t+1} + \lambda(\frac{\psi}{\alpha_p} - \frac{(\alpha_p - 1)}{\alpha_p})\tilde{y}_t{}^n, \quad (65)$$

$$y_t = c_t + \frac{\alpha\omega}{\sigma} s_t, \quad (66)$$

where $\tilde{y}_t{}^n = y_t - y_t^n$ is the output gap and $Z_y = \lambda(\psi/\alpha_p - (\alpha_p - 1)/\alpha_p)$. Plugging in the natural rate of output (9), condition (65) can also be written as

$$\pi_{H,t} = \beta E_t \pi_{H,t+1} + Z_y(y_t - \Gamma_0 + \Gamma_a a_t + \Gamma_* y_t^*)$$

$$\Rightarrow \hat{\pi}_{H,t} = \beta E_t \hat{\pi}_{H,t+1} + Z_y[\hat{y}_t - \Gamma_a \hat{a}_t - \Gamma_*(\hat{y}_t - \frac{1}{\sigma_\alpha}\hat{s}_t)],$$

$$\Rightarrow \hat{\pi}_{H,t} = \beta E_t \hat{\pi}_{H,t+1} + Z_y(1 - \Gamma_*)\tilde{y}_t + Z_y \frac{\Gamma_*}{\sigma_\alpha}\tilde{s}_t + f_1(e_t), \quad (67)$$

where $\tilde{x}_t = \hat{x}_t - \hat{x}_t^T$, and $\hat{x}_t^T$ denotes the target level of x_t. Target values are assumed to be functions of exogenous shock e_t that minimizes the loss when domestic inflation equals zero. For example, target levels of output and the terms of trade are $\hat{y}_t^T = q_y^e e_t$ and $\hat{s}_t^T = q_s^e e_t$. Specifically, target values of aggregate variables g_t^T equal $L_g^{-1} L_e e_t$. Assuming $e^R = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $e^L = \begin{bmatrix} 1 & 0 \end{bmatrix}$, we observe that

$$f_1(e_t) = [Z_y \Gamma_a e^L + Z_y(1 - \Gamma_*) q_y^e + \frac{Z_y \Gamma_*}{\sigma_\alpha} q_s^e] e_t.$$

Next, equation (66), in terms of the difference between the gap and the targeting gap, is given by

$$\tilde{y}_t = \frac{\alpha\omega + 1 - \alpha}{\sigma}\tilde{s}_t + f_2(e_t), \quad (68)$$

where

$$f_2(e_t) = (e^R - q^e_y + \frac{\alpha\omega + 1 - \alpha}{\sigma} q^e_s)e_t.$$

Notice that $\hat{\pi}^T_{H,t}$ equals zero implies that $\hat{p}^T_{H,t}$ equals $\hat{p}_{H,t-1}$. The gap between actual and target domestic price deviation $\tilde{p}_{H.t} = \hat{p}_{H,t} - \hat{p}_{H,t-1} = \hat{\pi}_{H,t}$. Denoting L_{ii} as the diagonal value of the matrix L_g where $i = \{1, 2, 3, 4\}$, the loss function (64) can also be written as

$$L^*_0 = E_0 \sum_{t=0}^{\infty} \beta^t u_c(\bar{C})\bar{C}[\frac{1}{2}(L_{11}\tilde{y}^2_t + L_{22}\tilde{c}^2_t + L_{44}\tilde{s}^2_t) + \frac{1}{2}(L_\pi + L_{33})\hat{\pi}^2_{H,t} + g'_t L^0_\pi \hat{\pi}_{H,t} + L_1(e'_t a^1_\pi + a^2_\pi)\hat{\pi}_{H,t}]$$

Using the fact that $\tilde{c} = ((1-\alpha)/\sigma)\tilde{s}_t$, the loss function can eventually be rearranged as

$$\begin{aligned} L^*_0 &= E_0 \sum_{t=0}^{\infty} \beta^t u_c(\bar{C})\bar{C}[\frac{1}{2}\Phi_y(\hat{y}_t - \hat{y}^T_t)^2 + \frac{1}{2}\Phi_s(\hat{s}_t - \hat{s}^T_t)^2 + \frac{1}{2}\Phi_\pi \hat{\pi}^2_{H,t} + g'_t L^0_\pi \hat{\pi}_{H,t} + L^1_\pi \hat{\pi}_{H,t}] + t.i.p + \mathcal{O}(||\xi^3||), \\ &= E_0 \sum_{t=0}^{\infty} \beta^t u_c(\bar{C})\bar{C}[\frac{1}{2}\Phi_y \tilde{y}^2_t + \frac{1}{2}\Phi_s \tilde{s}^2_t + \frac{1}{2}(\Phi_\pi + 2L^0_\pi(3,1))\hat{\pi}^2_{H,t} + (L^0_\pi(1,1)\tilde{y}_t + (L^0_\pi(2,1)\frac{1-\alpha}{\sigma} + L^0_\pi(4,1))\tilde{s}_t)\hat{\pi}_{H,t} \\ &+ L^0_\pi(4,1))\tilde{s}_t)\hat{\pi}_{H,t} + L^0_\pi(4,1))\tilde{s}_t)\hat{\pi}_{H,t} + L_1 e'_t(a^1_\pi + \Upsilon')\hat{\pi}_{H,t} + L_1 a^2_\pi \hat{\pi}_{H,t}] + t.i.p + \mathcal{O}(||\xi^3||), \quad (69) \end{aligned}$$

where

$$\Upsilon = L^0_\pi(1,1)q^e_y + (L^0_\pi(2,1)\frac{1-\alpha}{\sigma} + L^0_\pi(4,1))q^e_s + L^0_\pi(3,1)q^e_p,$$

$$\begin{bmatrix} q^e_y \\ q^e_c \\ q^e_p \\ q^e_s \end{bmatrix} = 2L^{-1}_g L_e,$$

$$\Phi_y = L_{11},$$

$$\Phi_s = L_{22}(\frac{1-\alpha}{\sigma})^2 + L_{44},$$

$$\Phi_\pi = L_\pi + L_{33}.$$

Finally, the social planner minimizes the loss function (69) subject to equation (67) and

(68). The first-order conditions are

$$\Delta\psi_{1,t} = \psi_{1,t} - \psi_{1,t-1} = [\Phi_\pi + 2L^0_\pi(3,1)]\hat{\pi}_{H,t} + L^0_\pi(1,1)\tilde{y}_t + \\ + [L^0_\pi(2,1)\frac{1-\alpha}{\sigma} + L^0_\pi(4,1)]\tilde{s}_t + L_1 e'_t(a^1_\pi + \Upsilon'), \tag{70}$$

$$-Z_y(1-\Gamma_*)\psi_{1,t} + \psi_{2,t} = \Phi_y\tilde{y}_t + L^0_\pi(1,1)\hat{\pi}_{H,t}, \tag{71}$$

$$-\frac{Z_y\Gamma_*}{\sigma_\alpha}\psi_{1,t} - \frac{\alpha\omega + 1 - \alpha}{\sigma}\psi_{2,t} = \Phi_s\tilde{s}_t + [L^0_\pi(2,1)\frac{1-\alpha}{\sigma} + L^0_\pi(4,1)]\hat{\pi}_{H,t}. \tag{72}$$

Combining equation (70), (71), and (72) by replacing $\psi_{1,t}$ and $\psi_{2,t}$, we obtain the optimal monetary policy as a targeting rule:

$$\mathcal{C}_y\tilde{y}_t + \mathcal{C}_s\tilde{s}_t + \mathcal{C}_\pi\hat{\pi}_{H,t} = \mathcal{C}^0_y\tilde{y}_{t-1} + \mathcal{C}^0_s\tilde{s}_{t-1} + \mathcal{C}^0_\pi\hat{\pi}_{H,t-1} + \mathcal{C}e_t \tag{73}$$

where

$$\begin{aligned}
\mathcal{C}_y &= L^0_\pi(1,1) + \frac{\mu_2\Phi_y}{\mu_1}, \\
\mathcal{C}_s &= L^0_\pi(2,1)\frac{1-\alpha}{\sigma} + L^0_\pi(4,1) + \frac{\Phi_s}{\mu_1}, \\
\mathcal{C}_\pi &= \Phi_\pi + 2L^0_\pi(3,1) + \frac{L^0_\pi(1,1) + [L^0_\pi(2,1)\frac{1-\alpha}{\sigma} + L^0_\pi(4,1)]}{\mu_1}, \\
\mathcal{C}^0_y &= \frac{\mu_2\Phi_y}{\mu_1}, \\
\mathcal{C}^0_s &= \frac{\Phi_s}{\mu_1}, \\
\mathcal{C}^0_\pi &= \frac{L^0_\pi(1,1) + [L^0_\pi(2,1)\frac{1-\alpha}{\sigma} + L^0_\pi(4,1)]}{\mu_1}, \\
\mathcal{C} &= -L_1(a^{1'}_\pi + \Upsilon), \\
\mu_2 &= \frac{\alpha\omega + 1 - \alpha}{\sigma}, \\
\mu_1 &= Z_y(1-\Gamma_*)\mu_2 + \frac{Z_y\Gamma_*}{\sigma_\alpha}.
\end{aligned}$$

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