# Reconciling ICP Benchmarks and Domestic Deflators: When Two Approaches Meet.

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This is Work in Progress. Draft Version: 3 May 2019. Please do not quote without authors permission.

**Abstract**

The paper presents a general method of interpolation/extrapolation that encompasses a number of alternatives proposed in the literature to reconcile the relationship between ICP Purchasing Power Parity (PPPs)'s Benchmarks and domestic deflators. We rely on well known statistical principles to derive a general reconciliation formula. While Diewert and Fox (2017) approach the reconciliation from an index number perspective, we show that the weights derived when their approach is written as a blended interpolation of PPPs (Diewert, 2018) is a special case of the general statistical form derived here. We present empirical results based on simulated data as well as actual data. The simulation provides an opportunity to study a number of special cases of the general formula. Using data at the basic heading level we compare the current PPP produced by EuroStat to those derived from the general framework. Finally we link currently available GDP level PPPs to the results derived in the paper.

# 1 Introduction

$PPP$ (Purchasing Power Parities) exchange rates and domestic inflation movements are intrinsically related. PPPs are international comparison prices and thus by definition relate domestic prices to those of a reference economy. Domestic deflators, such as consumer price indices (CPI) or GDP deflators, provide information on domestic price movements. Movements of PPPs over time are given by the relative movements in the domestic prices of each economy relative to the movements in the domestic prices of the reference economy.

The expression that provides the relationship between PPPs and domestic deflators is,

$$PPP_{it} = PPP_{i,t-1} \times IPD_{it} \tag{1}$$

where $IPD_{it} = \frac{Def_{i,[t-1,t]}}{Def_{1,[t-1,t]}}$, the ratio of the price change between $t-1$ and $t$ in country $i$ (domestic deflator for country $i$) to the change in prices for the same period in the reference economy which we set to be $i = 1$ without loss of generality to indicate that any country could be used as the reference. The reference country's PPP is by definition $PPP_{1t} \equiv 1$ for all $t$. It is standard practice to use the USA as the reference country and thus the PPPs are expressed in USD. We will let $t = 0, ..., T$ be the number of observed time periods and $i = 1, \ldots, N$ is the number of countries including the reference economy.

This paper presents a general method of interpolation/extrapolation that encompasses a number of alternatives proposed in the literature to reconcile this relationship. We show how standard statistical concepts provide a framework to write a reconciliation relationship as a weighted average of a forward and a backward prediction of the PPPs given the available information. This weighted average is an optimum estimator for this problem. Some of the concepts used here are common to the approach used by Rao, Rambaldi and Doran (2010). However, here we concentrate on the reconciliation problem and specifically we aim at deriving a general form of the weights. It is shown that the weights derived by the Diewert and Fox (2017) approach, when written as a blended interpolation of PPPs (Diewert, 2018), and those of the New Generation PWT (when written as a geometric average) are a special case of this general form. The concepts proposed in this note are general and can be applied at any level of aggregation from basic heading to GDP. An additional advantage of having a general form of the weights is that it allows us to study a number of alternative scenarios.

The rest of the paper is divided into four sections. Section 2 sets the notation and introduces all the key measurements. In this section it is assumed that PPPs are available for each country at every time period so that a general reconciling expression can be first derived. Section 3 provides a number of special cases that are derived by making alternative assumptions about the available PPP data and the measurement error associated with both PPPs and deflators data. Here we consider the case when PPPs are not available at each time period as well as the case when PPPs are assumed to be measured without error. We provide a number of results. Section 4 provides a number of empirical illustrations using simulated data, basic heading data and GDP level data. Section 5 concludes.

## 2 General Framework

It will be convenient to work using logarithmic transformations. We start by denoting the vector of logarithmic transformed PPPs for the $N$ countries at time period $t$, $t = 0, \ldots, T$, by $p_t = \ln(PPP_t)$, and by $c_t^o$ the $N \times 1$ vector with elements $c_{it}^o = \ln\left(\frac{Def_{i,[t-1,t]}}{Def_{1,[t-1,t]}}\right)$. The quantity $c_{it}^o$ is the natural logarithmic transformation of the $IPD_{it}$, and it is the *observed implicit growth rate* of country $i$'s PPP. When considering PPPs for GDP level, the observed implicit growth rate measures the change from $t-1$ to $t$ in the ratio of country $i$'s GDP deflator, to the change in the GDP deflator in the reference economy over the same period. To derive a general reconciling expression we allow both the PPPs and deflators to be measured with error. This provides a general framework whereby the cases of PPPs or IPDs being measure without error are special cases.

We introduce measurement error in the observed implicit growth rate by assuming the observed is the sum of the "true" rate, $c_t$, and $\eta_t$, a random term which is normal zero mean with variance, $V_{ct}$. Equation (2)

$$c_t^o = c_t + \eta_t \tag{2}$$

Similarly, we introduce a distinction between observed and true PPPs in equation (3). We denote the *observed log PPP* for period $t$ by $\tilde{p}_t$ and similarly to the case of the implicit growth rate, define it as the sum of the unobserved "true", $p_t$ and some measurement error, $\epsilon_t$, which we also assume normal with zero mean, variance, $V_{pt}$, and uncorrelated with $\eta_t$. This later assumption reflects our understanding that the compilation of PPPs for ICP is a separate exercise from that undertaken to produce national accounts

$$\tilde{p}_t = p_t + \epsilon_t. \tag{3}$$

We note here that this is in the same spirit as Summer and Heston (1988). The next version of this paper will show how their proposal is related to the approached considered here.

Taking logarithms of both sides of equation (1) we obtain

$$p_t = p_{t-1} + c_t$$

It then follows that if a PPP measurement is available at time $t-1$ for all $N-1$ countries that are not the reference, we can produce a prediction for time $t$ using the observed implicit growth rates constructed from available national accounts data. The notation "$t|t-1$" indicates the PPP information is from period $t-1$,

$$\widehat{p_{t|t-1}} = \tilde{p}_{t-1} + c_t^o. \tag{4}$$

The prediction $\widehat{p_{t|t-1}}$ is one within a number of possible conditional predictors that can be implemented depending on the available information at the time the prediction is made. In the next section we explore this in more detail and provide the basic elements to derive a general reconciliation expression which is a weighted average of two predictions.

## 2.1 Notation and Key Available Measurements

Statistically we can think about three key predictors that are relevant to the reconciliation problem. The first is a *forward* predictor, $E(p_{t|t-1}) = p^F_{t|t-1}$. This is based on what we know about the PPPs at $t-1$. Equation (4) is a forward predictor. Thus for example at $t=1$ and having observed a vector of PPPs at $t=0$, $\tilde{p}_0 = \ln(PPP_0)$, we would have

$$p^F_{1|0} = \tilde{p}_0 + c^o_1 \tag{5}$$

Suppose we have PPPs for each $i$ and each $t$, then we can construct a full panel of *forward* predictions. The forward predictor for $t=1$ is given by (5), and the remaining periods are similarly obtained,

$$p^F_{t|t-1} = \tilde{p}_{t-1} + c^o_t,\ t = 2, \ldots, T \tag{6}$$

We can also define a variance of this forward predictor, $P_{t|t-1}$. As the two parts of the sum are not correlated the covariance for $t = 1, \ldots, T$ is given by

$$P_{t|t-1} = Var(p^F_{t|t-1}) = Var(\tilde{p}_{t-1} + c^o_t) = V_{p,t-1} + V_{c,t}. \tag{7}$$

How can we relate $p^F_{t|t-1}$ to $\tilde{p}_t$? We know it is unlikely that $p^F_{t|t-1} = \tilde{p}_t$. In this case we both have a forward prediction available for period $t$ and then observe a PPP at $t$. This provides a second predictor of interest which provides the opportunity to combine the deflator movements and the observed PPPs. We label this predictor *updated forward predictor*, and can obtain it by applying a Lemma from the multivariate normal distribution given the assumptions we have made about $\eta_t$ and $\epsilon_t$(see for example Harvey (1989), App. to Ch 3). The straightforward application of the Lemma provides the following updated predictor for $t=1$

$$\begin{aligned} p^U_{1|1} &= p^F_{1|0} + \Omega_1(\tilde{p}_1 - p^F_{1|0}) \\ p^U_{1|1} &= (I - \Omega_1)p^F_{1|0} + \Omega_1 \tilde{p}_1 \end{aligned} \tag{8}$$

where,

$\Omega_1 = P_{1|0} V_{e1}^{-1}$

$V_{e1} = Var(\tilde{p}_1 - p^F_{1|0})$ and $P_{1|0}$ is defined in (7).

In general for $t = 1, \ldots, T$

$$p^U_{t|t} = (I - \Omega_t)p^F_{t|t-1} + \Omega_t \tilde{p}_t \tag{9}$$

$$Var(p^U_{t|t}) = P_{t|t} = P_{t|t-1} - P_{t|t-1}V^{-1}_{et}P_{t|t-1} \tag{10}$$

$$P_{t|t} = (I - \Omega_t)P_{t|t-1} \tag{11}$$

The expression in (9) is a weighted sum of the forward predictor and the observed PPP where the weights are functions of the variances of the measurement error in the implicit growth rates as well as the PPPs. The expression in (11) is the variance-covariance of the updated predictor. The $N \times N$ matrix of weights in (9) and (11) is defined as follows,

$$\Omega_t = P_{t|t-1}V^{-1}_{et} \tag{12}$$

which is a function of the conditional variance $P_{t|t-1}$, in (7), and the variance of the prediction error,

$$V_{et} = Var(\tilde{p}_t - p^F_{t|t-1}).$$

This general case where both data sources are allowed to be measured with error is similar to Rao et al (2010). They defined the variances of the measurement error to be, $V_{pt} = \sigma^2_\epsilon V^*_{pt}$ for PPPs and $V_{ct} = \sigma^2_\eta V^*_{ct}$ for the deflators. They show that under the stated assumptions the $V^*_{jt}$ $j = p, c$ matrices must take the following form for the system to preserve the invariance of PPPs to a change in the reference country (e.g. from the USA to the UK). Let $\sigma^2_{it}$ be the inverse of the reliability of the data of country $i$ at time $t$, then for the system of $N$ countries,

$$V^*_{jt} = \begin{bmatrix} 0 & 0 & \dots & & 0 \\ 0 & (\sigma^2_{2t} + \sigma^2_{1t}) & \sigma^2_{1t} & \dots & \sigma^2_{1t} \\ \vdots & \sigma^2_{1t} & \ddots & \vdots & \vdots \\ & \vdots & & & \sigma^2_{1t} \\ 0 & \sigma^2_{1t} & \dots & \sigma^2_{1t} & (\sigma^2_{Nt} + \sigma^2_{1t}) \end{bmatrix} \tag{13}$$

where, $\sigma_{it}^2$ is an inverse measure of the reliability of country's $i$ measurement of its prices. This form recognizes that country 1 is the reference and its PPP is 1 by definition, and thus it is a constraint to the system. However, this does not imply country 1 measures its prices without error (i.e., $\sigma_{1t}^2 \neq 0$ in general). Rao et al (2010) show that this form is a necessary condition for the invariance of the system to be maintained when constructing weighted estimates assuming both PPPs and growth rates are measured with error. $\sigma_\eta^2$ and $\sigma_\epsilon^2$ are constants of proportionality.

Finally we define a third predictor, $E(p_{t|T}) = p_{t|T}$, which is also a weighted sum and a harmonizing predictor once we have observed $T$ periods. Drawing from a number of works in the literature (Anderson and Moore (1979, Ch 7), Jazwinski (1970, pp 216-17) and Ansley and Kohn(1982)) the time series can be harmonized to account for all the information observed. This predictor is known in the filtering literature as a smoother (see Harvey(1990) section 3.6.1 for example), and with minor manipulations of a smoother's standard expression we can adapt it to our problem. Specifically, we can write the reconciliation formula as a weighted sum of a forward predictor, either or $p_{t|t}^F$ or $p_{t|t}^U$ (details in the next section) and a backward predictor as defined in (14)

$$p_t^B = \widehat{p_{t+1|T}} - c_{t+1}^o \tag{14}$$

where $\widehat{p_{t+1|T}}$ is a computed value of $E(p_{t+1|T})$. For instance, if a PPP is observed in period $T$ without error, the backward interpolated PPP for period $t$will be $p_\tau^B = \tilde{p}_T - \sum_{t=\tau+1}^T c_t^o$. That is, we subtract the observed deflator growth rates for periods $t = T, ..., \tau + 1$.

We are now ready to write the general expression for the reconciliation which we label with a superscript 'R' to indicate the expression is the reconciled PPP which combines the forward and backward predictors,

$$p_{t|T}^R = (I_N - \hat{\Psi}_t)p_{t|t}^U + \hat{\Psi}_t(\widehat{p_{t+1|T}} - c_{t+1}^o) \tag{15}$$

$$p_{t|T}^R = (I_N - \hat{\Psi}_t)p_{t|t}^U + \hat{\Psi}_t p_t^B \tag{16}$$

for $t = 1, \ldots, T$

where $I_N$ is the identity of size $N$, and the reconciliation weights have the general form,

$$\hat{\Psi}_t = P_{t|t}P_{t+1|t}^{-1} \tag{17}$$

To provide an example, (15) for $t = T - 1$ would be,

$$p_{T-1|T} = (I_N - \hat{\Psi}_{T-1})p^U_{T-1|T-1} + \hat{\Psi}_{T-1}(p_{T|T} - c^o_T)$$

and $p_{T|T} = p^U_{T|T}$ in the general case and $p_{T|T} = \tilde{p}_T$ if PPPs are not measured with error. We return to this case in the next section.

# 3 The Weights Under Alternative Assumptions

In this section we derive explicit forms of the reconciliation weights under alternative assumptions.

## 3.1 PPPs are only observed sparsely and assumed to be measured without error

We explore this case by assuming the first benchmark is observed at $t = 0$ and the second benchmark is observed at $t = T$. The reconciliation formula will be used to interpolate for periods $t = 1, \ldots, T - 1$. The $N \times 1$ vector of logarithmic transformed ICP benchmark PPPs are $\tilde{p}_0 = \ln(PPP_0)$ and $\tilde{p}_T = \ln(PPP_T)$ , respectively. We can construct the forward predictions for the entire sample. Since $E(\eta_t) = 0$ we have,

$$p^F_{1|0} = \tilde{p}_0 + c^o_1, \ t = 1 \tag{18}$$

$$p^F_{t|t-1} = p^F_{t-1|t-2} + c^o_t, \ t = 2, \ldots, T \tag{19}$$

The set of forward predictions are such that in the first time period we can use the vector of observed implicit growth rates $c^o_1$ to update the observed PPPs from the first benchmark, and then continue the updating by simply adding the corresponding growth rate until we observe the second benchmark at time $T$. This produces a complete time series of predicted PPPs for each of the $N$ countries. The uncertainty associated with the growth rates, $c^o_t$, which we denoted $\eta_t$, has covariance $V_{ct}$. With these defined we can establish the first result.

**Proposition 1.** *In the absence of a new ICP benchmark updating is not possible for periods* $t = 1, \ldots, T-1$. *Then under the stated assumptions it follows that the only feasible forward predictor and its variance-covariance are given by,*

$$E(p_{t|t}) = p^U_{t|t} = p^F_{t|t-1}$$

$$Var(p_{t|t}) = Var(p_{t|t-1}) = P_{t|t-1} = \sum_{\tau=1}^{t} V_{c\tau}$$

*Proof: The result follows from Durbin and Koopman (2012), Section 4.10 for filtering under missing information and* $Var(c^0_t = Var(\eta_t)$ *given the information.*

For $t = T$ the forward prediction can be updated as a new benchmark is available using expression (9). Then we can establish the following result.

**Lemma 1.** *The method delivers the expected end-point under when the ICP Benchmark for* $t = T$ *is measured without error. The PPP estimated by (9)is the observed ICP Benchmark since* $\Omega_T$ *is the identity.*

*Proof: Using expression (12)*

$$\Omega_T = P_{T|T-1} V^{-1}_{eT}$$

$$V_{eT} = Var(\tilde{p}_T - p^F_{T|T-1} - c_T)$$
$$V_{eT} = \sum_{t=1}^{T} V_{ct}$$

*By Proposition 1,*

$$\Omega_T = P_{T|T-1} V^{-1}_{eT} = \sum_{t=1}^{T} V_{et} [\sum_{t=1}^{T} V_{et}]^{-1} = I_N \tag{20}$$

*The updated estimate is then the observed Benchmark by replacing (20) in (11),*

$$
\begin{aligned}
p_{T|T} &= p^F_{T|T-1} + \tilde{p}_T - p^F_{T|T-1} \\
&= \tilde{p}_T
\end{aligned}
$$

We now turn to the reconciliation for the periods $T-1 \leq \tau \leq 1$. The expression is given in general by (15). In this it takes the form[1]

$$p^R_\tau = (I_N - \hat{\Psi}_\tau) p^F_{\tau|\tau-1} + \hat{\Psi}_\tau (\tilde{p}_T - \sum_{t=\tau+1}^{T} c_t) \tag{21}$$

where the standard expression for $\hat{\Psi}_\tau$ given the assumptions is,

$$\hat{\Psi}_\tau = P_{\tau|\tau} P^{-1}_{\tau|\tau-1} = (\sum_{t=1}^{\tau} V_{ct}) \left[ \sum_{t=1}^{\tau+1} V_{ct} \right]^{-1} \tag{22}$$

We note that the series of unobserved $\ln(PPP_\tau)$ are flow variables, and thus we use the following modified definition for the weights matrix, (22), so they reflect the relative contribution of the uncertainty of each missing period in relation to the overall uncertainty over the interval between the two observed benchmarks.

$$\hat{\Psi}_\tau = P_{\tau|\tau} P^{-1}_{T|T-1} = (\sum_{t=1}^{\tau} V_{ct}) \left[ \sum_{t=1}^{T} V_{ct} \right]^{-1} \tag{23}$$

We are now in a position to write one of the main results of the note which we state as a theorem with two Corollaries.

**Theorem 1.** *When the PPPs are measured without error, the interpolated PPPs are a weighted sum of a forward prediction,* $p^F_{\tau|\tau-1}$*, and backward prediction,* $p^B_\tau = \tilde{p}_T - \sum_{t=\tau+1}^{T} c_t$ *for all countries except the reference, and the weights are known functions of* $V_{ct}$ *(see (13))*

*Proof: for* $i = 1$ *it is true by Lemma 1.*

*For* $i > 1$*, by re-writing equation (21),*

[1] Standard Errors could be computed from the variance-covariance matrix $P^R_\tau = P_{\tau|\tau} + \hat{\Psi}_\tau (P_{\tau+1|\tau+1} - P_{\tau+1|\tau}) \hat{\Psi}'_\tau$

$$p_\tau^{R(N-1)} = (I_{N-1} - \hat{\Psi}_\tau^{(N-1)})p_{\tau|\tau-1}^{F(N-1)} + \hat{\Psi}_\tau^{(N-1)} p_\tau^{B(N-1)} \tag{24}$$

*where the superscript $(N-1)$ is to indicate this is the expression for the $i = 2, \ldots, N$ countries and thus it is the system obtained by eliminating the first row of $p_t^R$ which is zero for all $t$, and first row and column of $V_{ct}$ which are zero vectors.*

*The weights for each country $i$ are functions of all elements in $V_{ct}$ by definition (23) since $V_{ct}$ is not a diagonal matrix.*

**Corollary 1.** *When the reference country is assumed to measure growth rates without error then $V_{ct}$ becomes a diagonal as $\sigma_{1t}^2 = 0$. In this case the $\hat{\Psi}_\tau$ will also be diagonal. For $i > 1$ (23) simplifies to*

$$p_{i,\tau}^s = (1 - \hat{\Psi}_{i,\tau})p_{i,\tau|\tau}^F + \hat{\Psi}_{i,\tau}(p_{i,\tau+1}^s - c_{i,\tau+1}) \tag{25}$$

*and*

$$\hat{\Psi}_{i,\tau} = \left(\sigma_\eta^2 \sum_{t=1}^{\tau} \sigma_{it}^2\right) / \left(\sigma_\eta^2 \sum_{t=1}^{T} \sigma_{it}^2\right) = \left(\sum_{t=1}^{\tau} \sigma_{it}^2\right) / \left(\sum_{t=1}^{T} \sigma_{it}^2\right) \tag{26}$$

*which shows the weights assigned to country $i$ are not functions of the measurement error of the reference country or any other country in the system.*

**Corollary 2.** *When the reference country is assumed to measure growth rates without error, $\sigma_{1t}^2 = \sigma_1^2 = 0$ , and all other the elements of $V_{ct}$ do not vary overtime, i.e. $\sigma_{it}^2 = \sigma_i^2$, the weights reduce to the blended PPP estimates in Diewert (2018) equation (26) and thus equivalent to the Diewert and Fox (2017) approach (see Diewert (2018), and to the New Generation PWT weighting scheme (Feenstra et al, 2015) when written as a geometric average. This holds even if the variances are different across countries (heteroscedastic).*

$$\begin{aligned}\hat{\Psi}_{i\tau} &= \left(\sum_{t=1}^{\tau} \hat{\sigma}_i^2\right) / \left(\sum_{t=1}^{T} \hat{\sigma}_i^2\right) \\ &= \frac{\tau \hat{\sigma}_i^2}{T \hat{\sigma}_i^2} = \frac{\tau}{T}\end{aligned}$$

### 3.2 PPPs are only observed sparsely and assumed to be measured with error

We now assume benchmarks are measured with error and observed sparsely. Again the first benchmark is observed at $t = 0$ and the second benchmark is observed at $t = T$. The $N \times 1$ vector of logarithmic transformed ICP benchmark PPPs are $\tilde{p}_0$ and $\tilde{p}_T$, respectively. These are assumed to be observations of the true but unobserved latent variables $p_0$ and $p_T$.

$$\tilde{p}_0 = p_0 + \epsilon_0 \tag{27}$$

$$\tilde{p}_T = p_T + \epsilon_t \tag{28}$$

$\epsilon_t \sim N(0, V_{pt}),\ t = 0, T$

In this case the forward predicting equations are given by

For $t = 1$

$$p^F_{1|0} = \tilde{p}_0 + c^o_1$$

$$P_{1|0} = V_{p0} + V_{c1}$$

For $t = 2, \ldots, T$ the forward predictor, $p^F_{T|T-1}$, is given by (19). There is no benchmark information for $t = 1, \ldots, T-1$, and thus the updated forward predictor is not applicable. there is no updating for these time periods. At time $t = T$, the second benchmark provides information to update $p^F_{T|T-1}$ and $P_{T|T-1}$,

$$p^U_{T|T} = p^F_{T|T-1} + \Omega_T(\tilde{p}_T - p^F_{T|T-1} - c_T) \tag{29}$$

$$P_{T|T-1} = P_{T-1} + V_{cT} \tag{30}$$

$$V_{eT} = P_{T|T-1} + V_{pT}$$

$$\Omega_T = P_{T|T-1}V^{-1}_{eT} \tag{31}$$

$$P_{T|T} = P_{T|T-1} - P_{T|T-1}V^{-1}_{eT}P_{T|T-1} \tag{32}$$

Equation (29) corresponds to (15) with $\hat{\Psi}_T = \Omega_T$

$$p_{T|T} = p^U_{T|T} \tag{33}$$

$$p_{T|T} = \hat{\Psi}_T(\tilde{p}_T - c_T) + (I - \hat{\Psi}_T)p^F_{T|T-1} \tag{34}$$

For $t = 1, \ldots, T-1$ the reconciliation formula ($E(p_{t|T})$ expression ) is obtained from the expression

$$p^R_t = (1 - \hat{\Psi}_t)p^F_{t|t-1} + \hat{\Psi}_t(p^R_{t+1} - c_t) \tag{35}$$

In this case we obtain the following results

**Result 1.** Lemma 1 not longer applies and thus the interpolated PPPs for the periods between the two benchmarks, $p^R_\tau$ $\tau = 1, \ldots, T-1$, are a function of the measurement errors in both benchmarks ($V_{p0}$ and $V_{pT}$).

**Result 2.** The weights, (17), $\hat{\Psi}_\tau$ for $\tau = 1, \ldots, T-1$, are only a function of the measurement error of the first benchmark, $V_{p0}$, and thus will not change independently of whether the measurement error in the second benchmark differs from that of the first.

**Result 3.** Corollary 2 does not hold when the first benchmark is measured with error. To show this assume $V_{pt}$ and $V_{ct}$ are time invariant and proportional to the same diagonal matrix.

The weights' expression in (17) for $\tau = 1, \ldots, T-1$ is

$$\begin{aligned}
\hat{\Psi}_\tau &= P_{t|t} P_{t+1|t}^{-1} \\
&= (V_{p0} + \sum_{t=1}^{\tau} V_{ct})(V_{p0} + \sum_{t=1}^{T} V_{ct})^{-1} \\
&= (\sigma_\epsilon^2 V_{0t}^* + \sigma_\eta^2 \sum_{t=1}^{\tau} V_{ct}^*)(\sigma_\epsilon^2 V_{0t}^* + \sigma_\eta^2 \sum_{t=1}^{T} V_{ct}^*)^{-1}
\end{aligned}$$

Simplifying the expression by $V_{0t}^* = V_{ct}^* = V$ for all $t = 0, 1, \ldots, T-1$ provides the following result, which makes it clear that the expression becomes that in Corollary 2 when $\sigma_\epsilon^2 = 0$ (i.e., under no measurement error in the PPPs)

$$\begin{aligned}
\hat{\Psi}_\tau &= (\sigma_\epsilon^2 V + \sigma_\eta^2 \tau V)(\sigma_\epsilon^2 V + \sigma_\eta^2 T V)^{-1} \\
&= [(\sigma_\epsilon^2 + \sigma_\eta^2 \tau) V][(\sigma_\epsilon^2 + \sigma_\eta^2 T) V]^{-1} \\
&= (\sigma_\epsilon^2 + \sigma_\eta^2 \tau)(\sigma_\epsilon^2 + \sigma_\eta^2 T)^{-1}
\end{aligned}$$

# 4 Empirical Results (incomplete)

## 4.1 Simulated Example

### 4.1.1 Basic Data

Tables (1) and (2) presents the basic data for this example. Here we have five countries, $N = 5$, and country 1 is the reference. We have four time periods, $t = 0, \ldots, 3$. In 2 we provide the assumed variances of measurement errors for countries $i = 2, \ldots, 5$. The variances for country one will be changed under alternative scenarios considered below.

Table 1: Simulated Data. Five Countries and four time periods

| Country | $PPP_0$ | $\frac{GDPDef_{i,[0,1]}}{GDPDef_{1,[0,1]}}$ | $\frac{GDPDef_{i,[1,2]}}{GDPDef_{1,[1,2]}}$ | $\frac{GDPDef_{i,[2,3]}}{GDPDef_{1,[2,3]}}$ | $PPP_3$ |
|---|---|---|---|---|---|
| 1 | 1 | | | | 1 |
| 2 | 0.7 | 1.0400 | 1.0200 | 1.0500 | 0.8 |
| 3 | 3.4 | 1.0500 | 1.0500 | 1.0200 | 4.3 |
| 4 | 89 | 1 | 1.0200 | 1.0100 | 91 |
| 5 | 14.67 | 1.0300 | 1.0200 | 1.0100 | 16 |
| Five countries, four time periods, PPPs observed at $t = 0$ and $t = T = 3$ | | | | | |

Table 2: Assumed Variances of Measurement Errors

| | $V_{ct}$ | $V_{pt}$ |
|---|---|---|
| $t = 0$ | | $\begin{bmatrix} 0.002+\sigma_{10}^2 & \sigma_{10}^2 & \sigma_{10}^2 & \sigma_{10}^2 \\ \sigma_{10}^2 & 0.004+\sigma_{10}^2 & \sigma_{10}^2 & \sigma_{10}^2 \\ \sigma_{10}^2 & \sigma_{10}^2 & 0.001+\sigma_{10}^2 & \sigma_{10}^2 \\ \sigma_{10}^2 & \sigma_{10}^2 & \sigma_{10}^2 & 0.05+\sigma_{10}^2 \end{bmatrix}$ |
| $t = 1$ | $\begin{bmatrix} 0.02+\sigma_{11}^2 & \sigma_{11}^2 & \sigma_{11}^2 & \sigma_{11}^2 \\ \sigma_{11}^2 & 0.05+\sigma_{11}^2 & \sigma_{11}^2 & \sigma_{11}^2 \\ \sigma_{11}^2 & \sigma_{11}^2 & 0.01+\sigma_{11}^2 & \sigma_{11}^2 \\ \sigma_{11}^2 & \sigma_{11}^2 & \sigma_{11}^2 & 0.08+\sigma_{11}^2 \end{bmatrix}$ | |
| $t = 2$ | $\begin{bmatrix} 0.03+\sigma_{12}^2 & \sigma_{12}^2 & \sigma_{12}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & 0.04+\sigma_{12}^2 & \sigma_{12}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{12}^2 & 0.02+\sigma_{12}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{12}^2 & \sigma_{12}^2 & 0.09+\sigma_{12}^2 \end{bmatrix}$ | |
| $t = 3$ | $\begin{bmatrix} 0.021+\sigma_{13}^2 & \sigma_{13}^2 & \sigma_{13}^2 & \sigma_{13}^2 \\ \sigma_{13}^2 & 0.049+\sigma_{13}^2 & \sigma_{13}^2 & \sigma_{13}^2 \\ \sigma_{13}^2 & \sigma_{13}^2 & 0.01+\sigma_{13}^2 & \sigma_{13}^2 \\ \sigma_{13}^2 & \sigma_{13}^2 & \sigma_{13}^2 & 0.10+\sigma_{13}^2 \end{bmatrix}$ | $\begin{bmatrix} 0.001+\sigma_{13}^2 & \sigma_{13}^2 & \sigma_{13}^2 & \sigma_{13}^2 \\ \sigma_{13}^2 & 0.001+\sigma_{13}^2 & \sigma_{13}^2 & \sigma_{13}^2 \\ \sigma_{13}^2 & \sigma_{13}^2 & 0.001+\sigma_{13}^2 & \sigma_{13}^2 \\ \sigma_{13}^2 & \sigma_{13}^2 & \sigma_{13}^2 & 0.001+\sigma_{13}^2 \end{bmatrix}$ |
| | $i = 2, \ldots, 5$ | |

#### 4.1.2 Scenarios

Three scenarios are run by choosing alternative values for the reference country's inverse reliability.

1. **PPPs Measured without error.** Reference country deflators are highly reliable.

   **Variance of Country 1** (inverse reliability measure):

   $\sigma_{11}^2 = 0.005$, $\sigma_{12}^2 = 0.004$, and $\sigma_{13}^2 = 0.008$;

2. **PPPs Measured without error.** Reference country deflators are not as reliable. I

   **Variance of Country 1** (inverse reliability measure):

   $\sigma_{11}^2 = 0.05$; $\sigma_{12}^2 = 0.04$; $\sigma_{13}^2 = 0.08$.

3. **PPPs Measured with error.** Reference country measurements are highly reliable.

   **Variance of Country 1** (inverse reliability measure) -GROWTH RATES

   $\sigma^2_{11} = 0.005$, $\sigma^2_{12} = 0.004$, and $\sigma^2_{13} = 0.008$;

   **Variance of Country 1** (inverse reliability measure) - PPPs

   $\sigma^2_{10} = 0.001$, $\sigma^2_{13} = 0.001$

The results are presented in Tables (3)-(5).

Table 3: Scenario 1- PPP no error, country 1 reliable

| $t$ | **0** | **1** | | | | | **2** | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | $PPP$ | $PPP^F$ | $PPP^R$ | $\hat{\Psi}$ | $PPP^{DF}$ | $\hat{\Psi}^{DF}$ | $PPP^F$ | $PPP^R$ | $\hat{\Psi}$ | $PPP^{DF}$ | $\hat{\Psi}^{DF}$ |
| 1 | **1** | 1 | 1 | | 1 | | 1 | 1 | | 1 | |
| 2 | **0.70** | 0.73 | 0.73 | **0.28** | 0.73 | **0.33** | 0.74 | 0.75 | **0.68** | 0.76 | **0.67** |
| 3 | **3.40** | 3.57 | 3.67 | **0.36** | 3.66 | **0.33** | 3.75 | 4.04 | **0.64** | 4.05 | **0.67** |
| 4 | **89.00** | 89.00 | 88.85 | **0.26** | 88.85 | **0.33** | 90.78 | 90.06 | **0.70** | 90.33 | **0.67** |
| 5 | **14.67** | 15.11 | 15.18 | **0.30** | 15.20 | **0.33** | 15.41 | 15.67 | **0.63** | 15.70 | **0.67** |

| $t$ | **3** | |
|---|---|---|
| $N$ | $PPP^F$ | $PPP$ |
| 1 | 1 | **1** |
| 2 | 0.78 | **0.80** |
| 3 | 3.82 | **4.30** |
| 4 | 91.69 | **91.00** |
| 5 | 15.57 | **16.00** |

$PPP$: Observed PPP
$PPP^F$: Forward Predictor using implicit deflators
$PPP^R$: harmonized weights $\hat{\Psi}$ based on full $V_{ct}$
$PPP^{DF}$: harmonized weights $\hat{\Psi}^{DW}$ are Diewert and Fox(2017) and PWT9
**$\hat{\Psi}^P$ , $\hat{\Psi}$ are weights on the backward predictor**

Table 4: Scenario 2- PPP no error, country 1 less reliable

| $t$ | 0 | 1 | | | | | 2 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | $PPP$ | $PPP^F$ | $PPP^R$ | $\hat{\Psi}$ | $PPP^{DF}$ | $\hat{\Psi}^{DF}$ | $PPP^F$ | $PPP^R$ | $\hat{\Psi}$ | $PPP^{DF}$ | $\hat{\Psi}^{DF}$ |
| 1 | 1 | 1 | **1** | | 1 | | 1 | **1** | | 1 | |
| 2 | 0.70 | 0.73 | **0.73** | **0.28** | 0.73 | 0.33 | 0.74 | **0.75** | **0.66** | 0.76 | 0.67 |
| 3 | 3.40 | 3.57 | **3.66** | **0.35** | 3.66 | 0.33 | 3.75 | **4.04** | **0.63** | 4.05 | 0.67 |
| 4 | 89.00 | 89.00 | **88.81** | **0.27** | 88.85 | 0.33 | 90.78 | **89.87** | **0.65** | 90.33 | 0.67 |
| 5 | 14.67 | 15.11 | **15.18** | **0.30** | 15.20 | 0.33 | 15.41 | **15.65** | **0.62** | 15.70 | 0.67 |

| $t$ | 3 | |
|---|---|---|
| $N$ | $PPP^F$ | $PPP$ |
| 1 | 1 | 1 |
| 2 | 0.78 | 0.80 |
| 3 | 3.82 | 4.30 |
| 4 | 91.69 | 91.00 |
| 5 | 15.57 | 16.00 |

$PPP$: Observed PPP
$PPP^F$: Forward Predictor using implicit deflators
$PPP^R$: harmonized weights $\hat{\Psi}$ based on full $V_{ct}$
$PPP^{DF}$: harmonized weights $\hat{\Psi}^{DF}$ are Diewert and Fox(2017) and PWT9
**$\hat{\Psi}^{DF}$ , $\hat{\Psi}$ are weights on the backward predictor**

Table 5: Scenario 3- PPP with error, country 1 reliable

| $t$ | 0 | | | 1 | | | | 2 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $N$ | $PPP$ | $PPP^R$ | $\hat{\Psi}$ | $PPP^R$ | $\hat{\Psi}$ | $PPP^{DF}$ | $\hat{\Psi}^{DF}$ | $PPP^R$ | $\hat{\Psi}$ | $PPP^{DF}$ | $\hat{\Psi}^{DF}$ |
| 1 | 1 | **1** | | **1** | | 1 | | **1** | | 1 | |
| 2 | 0.70 | **0.73** | **0.03** | **0.73** | **0.31** | 0.73 | 0.33 | **0.75** | **0.69** | 0.76 | 0.67 |
| 3 | 3.40 | **3.57** | **0.03** | **3.67** | **0.37** | 3.66 | 0.33 | **4.04** | **0.65** | 4.05 | 0.67 |
| 4 | 89.00 | **89.00** | **0.03** | **88.85** | **0.28** | 88.85 | 0.33 | **90.07** | **0.71** | 90.33 | 0.67 |
| 5 | 14.67 | **15.11** | **0.15** | **15.21** | **0.40** | 15.20 | 0.33 | **15.68** | **0.68** | 15.70 | 0.67 |

| $t$ | 3 | | |
|---|---|---|---|
| $N$ | $PPP^R$ | $\hat{\Psi}$ | $PPP$ |
| 1 | **1** | | 1 |
| 2 | **0.80** | **0.98** | 0.80 |
| 3 | **4.29** | **0.99** | 4.30 |
| 4 | **90.98** | **0.97** | 91.00 |
| 5 | **15.99** | **1.00** | 16.00 |

$PPP$: Observed PPP

$PPP^F$: Forward Predictor using implicit deflators

$PPP^R$: harmonized weights $\hat{\Psi}$ based on full $V_{ct}$ and $V_{pt}$

$PPP^{DF}$: harmonized weights $\hat{\Psi}^{DF}$ are PWT9

**$\hat{\Psi}^{DF}$ , $\hat{\Psi}$ are weights on the backward predictor**

## 4.2 EuroStat Basic Heading Data

In this section we present an example using basic heading data for countries in the EuroStat area. In this exercise we have PPPs for two years, 2005 and 2011. The domestic price movements are measured by the corresponding basic heading CPI. Table 6 presents the results for two countries for the Basic Heading: Flour and Cereals using Germany as the reference economy. The PPPs are treated as measured without error. The column labelled $PPP^R$ presents the interpolation obtained using the expression in (24) where the elements of the $V_{ct}$ matrix, $\sigma^2_{it}$, are measured using the inverse of each country's GDP per capita converted to USD using exchange rates and in constant prices of 2005. The column labelled $PPP^{DF}$ are obtained using the Diewert and Fox (2018) weights.

Table 6: Basic Heading Example

| Flour and Cereals | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Austria | | | | | Bulgaria | | | | |
| YEAR | $PPP$ | $PPP^F$ | EuroStat | $PPP^R$ | $PPP^{DF}$ | $PPP$ | $PPP^F$ | EuroStat | $PPP^R$ | $PPP^{DF}$ |
| **2005** | **1.1265** | **1.1265** | **1.1265** | **1.1265** | **1.1265** | **1.1553** | **1.1553** | **1.1553** | **1.155** | **1.155** |
| 2006 | | **1.07896** | **1.0667** | **1.0884** | **1.0840** | | **0.9612** | **1.0782** | **0.990** | **0.972** |
| 2007 | | **1.04743** | **1.0810** | **1.0919** | **1.0774** | | **0.8829** | **1.3165** | **1.004** | **0.943** |
| 2008 | | **1.02763** | **1.0349** | **1.1455** | **1.1185** | | **0.8952** | **1.4513** | **1.215** | **1.089** |
| 2009 | | **1.02354** | **1.0403** | **1.2478** | **1.2125** | | **0.9098** | **1.3940** | **1.530** | **1.347** |
| 2010 | | **1.00791** | **1.0410** | **1.3248** | **1.2996** | | **0.9063** | **1.3782** | **1.779** | **1.632** |
| **2011** | **1.3645** | **1.00581** | **1.3645** | **1.3645** | **1.3645** | **1.917** | **0.9461** | **1.9171** | **1.917** | **1.917** |

Note: The reference country is Germany

BH Data and Deflator(HCPI) data from Rao, Inklaar and Rambaldi (2018)- TAG meeting

$V_{ct}$: Inversely proportional to country's GDP per capita in constant USD using exchange rates.

The results would indicate EuroStat is using a forward predictor to interpolate between benchmarks.

# 5 How Publicly Available GDP Level PPPs Relate to the General Framework (incomplete)

In this section we compare available interpolated PPPs for GDP that are produced under alternative assumptions. These are

**Original** PWT: GDP is built from components's PPPs. Components PPPs are anchored on the last available ICP benchmark (assumed without error). For participating countries the observed component PPP and equation (1)is used to extrapolate backwards and forwards. For non-participating countries a cross sectional price level regression for the benchmark year is used to obtain a predicted PPP for each component and then equation (1)is used to extrapolate backwards and forwards. Some adjustments based on knowledge of individual countries were made. Summer and Heston (1988) proposed a "Consistentization" approach. This approach is in the spirit of what this paper presents as both PPPs and growth rates are treated as stochastic and measured with error. The next version of this paper will show the link more specifically.

**New** Generation PWT : PPPs assumed to be measured without error (although 2005 has been recomputed using 2011 methodology). Interpolation between ICP measurements using Corollary 2 weights, but applied to the PPP's levels rather than the logarithmic transformation. Deflator movements used to extrapolate.

**UQICDv2:** The reconciled predictor is given (15). The resulting PPPs is a geometric weighing of an updated forward predictor and a backward predictor. ICP and OECD benchmarks are used when available and predictions from a panel price level regression for non-participating countries and non-benchmark years are used when benchmarks are not available. The regression error structure is independent of the ICP/OECD measurement error. Deflator movements are used to compute the forward predictor, and PPP (ICP/OECD or regression prediction) are used to construct the updated forward predictor. The updated predictor for the last point in the sample, $T$, is the starting PPP to construct the backward predictor. All are treated with error.

Table 7: GDP Level PPP

| | China | | | | | India | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| YEAR | ICP | PWT71 | PWT91[a] | UQICD[b] | SE(UQICD) | ICP | PWT71 | PWT91[a] | UQICD[b] | SE(UQICD) |
| **2005** | 3.450 | 3.003 | 2.522 | 3.201 | 0.147 | 14.67 | 13.682 | 12.654 | 13.850 | 0.904 |
| 2006 | | 3.056 | 2.636 | 3.100 | 0.207 | | 14.092 | 12.837 | 12.741 | 1.249 |
| 2007 | | 3.153 | 2.963 | 3.141 | 0.229 | | 14.391 | 12.845 | 12.486 | 1.338 |
| 2008 | | 3.327 | 3.249 | 3.236 | 0.232 | | 14.747 | 13.260 | 12.953 | 1.392 |
| 2009 | | 3.266 | 3.252 | 3.106 | 0.209 | | 15.613 | 14.170 | 13.344 | 1.363 |
| 2010 | | 3.370 | 3.296 | 3.236 | 0.185 | | 16.839 | 14.486 | 14.252 | 1.272 |
| **2011** | 3.506 | | 3.488 | 3.448 | 0.127 | 15.109 | | 14.837 | 15.151 | 0.902 |

[a]Backed out from series "pl_gdpo" https://www.rug.nl/ggdc/productivity/pwt/

[b] http://uqicd.economics.uq.edu.au

## 6 Conclusion

The ICP commenced as a research exercise. It has evolved into a program of measuring prices across countries sporadically. To date each benchmark exercise is conducted in isolation. Its future is likely to view the exercise as one with a cross-country as well as a time dimension. We provide a framework that integrates both dimensions and can be implemented at several aggregation levels.

Using well established statistical principles these two dimensions can be combined to make use of all the information in an efficient way. We show the basic PPP updating relationship can be used to set up a general framework that can be related to a number of existing interpolation/extrapolation methods previously proposed for the problem. The empirical section uses a simulation to illustrate some of the results. The expressions derived are used to study the PPPs produced at basic headings by EuroStat. We compare GDP level PPPs constructed under some of the alternative assumptions made in the theoretical sections of the paper.

# References


[1] Anderson B. D. O and J. B Moore (1979), *Optimal Filtering.* Englewood Cliffs, N.J.: Prentice Hall.

[2] Ansley and Kohn (1982), "Geometrical Derivation Of The Fixed Interval Smoothing Algorithm". *Biometrika* , 9, 486-7.

[3] Diewert, W. E. (2018), "Alternative Methods for Interpolating PPPs". Discussion Paper 18-07, School of Economics, University of British Columbia.

[4] Diewert, W.E. and K.J. Fox (2017), "Output Growth and Inflation Across Space and Time", *EURONA* 2017:1, 7-40. https://ec.europa.eu/eurostat/cros/system/files/euronaissue1-2017-art1.pdf.

[5] Durbin and Koopman (2012), T*ime Series Analysis by State Space Methods, Second Edition.* Oxford University Place. DOI:10.1093/acprof:oso/9780199641178.001.0001

[6] Feenstra, R.C., R. Inklaar and M.P. Timmer (2015), "The Next Generation of the Penn World Table", American Economic Review 105, 3150-3182.

[7] Harvey, A. C. (1990), *Forecasting, Structural Time Series Models and the Kalman Filter.* Cambridge University Press.

[8] Jazwinski A. (1970). *Stochastic Processes and Filtering Theory*, Mathematics in Science and Engineering, Volume 64. Academic Press

[9] Rao, Rambaldi and Doran (2010), Extrapolation Of Purchasing Power Parities Using Multiple Benchmarks And Auxiliary Information: A New Approach, *Review of Income and Wealth Series* 56, Special Issue 1.

[10] Summer and Heston (1984), "New Set Of International Comparisons Of Real Product And Price Levels Estimates For 130 Countries, 1950-1985," *The Review of Income and Wealth*