Multilateral Sato-Vartia Index for International Comparisons of Prices and Real Expenditures

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Extended Abstract

Motivated by the presentations on Sato-Vartia indexes in the last two annual International Comparisons Conference, we explore the problem of constructing multilateral Sato-Vartia index numbers which satisfy transitivity. Transitivity is a consistency requirement which is not satisfied by most of the popular index numbers such as the Fisher, Tornqvist, Walsh and other index number formulae. The International Comparison Program (ICP) uses the Gini-Elteto-Koves-Szulc (GEKS) method which builds a transitive set of comparisons from a matrix of bilateral comparisons based on the Fisher index. The GEKS method is designed to ensure a high degree of *characteristicity* while satisfying transitivity – thus the GEKS comparisons between pairs of countries deviate the least (logarithmic least squares) from the corresponding binary comparisons. Though the GEKS procedure used in the ICP is based on Fisher binary indexes which have attractive economic theoretic properties and are superlative, we do not have economic theory that justifies GEKS (Neary, 2004).

In this paper, we explore two strands of the problem of constructing transitive Sato-Vartia index numbers. First, we explore and propose several ways of constructing transitive multilateral price comparisons anchored on the binary Sato-Vartia index. Secondly, we pursue the possibility of constructing multilateral price comparisons based on Constant Elasticity of Substitution (CES) preferences. In both cases, empirical implementation of the proposed indices is based on ICP data for the OECD price and real expenditure comparisons for Household Consumption for the benchmark years 2011 and 2014.

1. Sato-Vartia Index with Transitivity

The starting point for this work is the original paper by Sato (1976) where he derives a log-change index number that satisfies the factor reversal test. We use the following notation throughout this paper.

$\left\{ p_{ij}, q_{ij} : i = 1,2,...,N;\ j = 1,2,...,M \right\}$ - price and quantity, respectively, of *i-th* commodity in *j-th* country

$\left\{ P_{jk}, Q_{jk}, E_{jk} : j,k = 1,2,...,M \right\}$ - respectively price, quantity and value indexes for country *k* with country *j* as the base country. The log-change index numbers used in Sato (1976) defined as:

$$\ln P_{jk} = \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{p_{ik}}{p_{ij}}\right),\ \ln Q_{jk} = \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{q_{ik}}{q_{ij}}\right), \sum_{i=1}^{N} \phi_i^{jk} = 1, \phi_i^{jk} \geq 0 \qquad (1)$$

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[2] This study was initiated and a part of the work was undertaken when Rao was visiting Hitotsubashi Institute for Advanced Study, Hitotsubashi University during September-Novermber, 2018.

The value index: $E_{jk} = \sum_{i=1}^{N} p_{ik}q_{ik} \Big/ \sum_{i=1}^{N} p_{ij}q_{ij}$. Expenditure shares are denoted by: $w_{ij} = \dfrac{p_{ij}q_{ij}}{\sum_{i=1}^{N} p_{ij}q_{ij}}$.

If the price and quantity indexes in (1) satisfy the factor reversal test which stipulates that the product of price and quantity indexes (computed using the same formula) must equal the value index, then the weights ϕ_i^{jk} must satisfy the following condition:

$$\ln P_{jk} + \ln Q_{jk} - \ln E_{jk} = 0 \Rightarrow \sum_{i=1}^{N} \phi_i^{jk} \left(\frac{w_{ik}}{w_{ij}} \right) = 0. \tag{2}$$

Sato (1976) choses the following weights ϕ_i^{jk} that satisfy (2):

$$\phi_i^{jk} = \frac{\dfrac{w_{ik} - w_{ij}}{\ln w_{ik} - \ln w_{ij}}}{\sum_{i=1}^{N} \left(\dfrac{w_{ik} - w_{ij}}{\ln w_{ik} - \ln w_{ij}} \right)} \quad i = 1, 2, ..., N \tag{3}$$

These weights are non-negative and they sum to unit. From (1) and (3) it is evident that the matrix of Sato-Vartia indexes for all bilateral comparisons general fails transitivity.

We consider three alternative approaches to the construction of Transitive Sato-Vartia indices. The first approach is to apply GEKS formula to bilateral S-V indices; the second approach is to construct transitive indices that have weights that are closes to the weights used in bilateral S-V indexes; and the third approach tries to maintain the core principle of S-V index, viz., satisfying the factor reversal test. Progress under these approaches is discussed below;

Approach 1: GEKS

Under this approach we generate transitive indices that are closed to the matrix of binary S-V indices using the GEKS framework. We have

$$P_{jk}^{SV-GEKS} = \prod_{l=1}^{M} \left[P_{jl}^{SV} \times P_{lk}^{SV} \right]^{1/M} \quad where \ P_{jk}^{SV} = \prod_{i=1}^{N} \left[\frac{p_{ik}}{p_{ij}} \right]^{\phi_i^{jk}} \tag{4}$$

Where the weights ϕ_i^{jk} are those defined in (3).

Approach 2: Use weights closest to the weights in SV binary index

We first establish the structure of the SV index that satisfies transitivity (Result 1 below). Following on from this result, we consider different ways of generating ϕ_i which are closest to the ϕ_i^{jk} using least squares criteria. We derive expressions for unweighted and weighted least squares approach.

Result 1: The Sato-Vartia index (price or quantity) index defined in (1) satisfies transitivity if and only if the weights in (3) are identical to all pairs of countries, i.e.,

$$\phi_i^{jk} = \phi_i \quad for\ all\ \ j,k$$

Then the price index in (1) is of the form:

$$P_{jk} = \prod_{i=1}^{N}\left(\frac{p_{ik}}{p_{ij}}\right)^{\phi_i} \quad for\ all\ \ j,k = 1,2,...,M \tag{5}$$

This result is consistent with an important theorem by Funke, Hecker and Voeller (1979).

From Result 1 it is clear that we need a set of weights $\left\{\phi_i : i = 1,2,...,N\ such\ that\ \phi_i \geq 0,\ \sum_{i=1}^{N}\phi_i = 1\right\}$. In this paper we propose different criteria for generating these weights where the criteria are designed to maintain characteristicity so that the resulting transitive weights, ϕ_i, are as close as possible to the observed non-transitive S-V weights, ϕ_i^{jk}.

Result 2: Suppose we have a set of observed weights, ϕ_i^{jk} for all *i,j and k*. Then the optimum set of weights $\hat{\phi}_i$ that minimizes the objective function

$$\sum_{k=1}^{M}\sum_{j=1}^{M}\sum_{i=1}^{N}\frac{1}{2}\left(\phi_i^{j,k} - \phi_i\right)^2 \text{ subject to } \sum_{i=1}^{N}\phi_i = 1 \tag{6}$$

are given by:

$$\hat{\phi} = \overline{\phi}_i = \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)}{M^2} \text{ for all i.} \tag{7}$$

Proof is straightforward. It is easy to show that these are non-negative and add up to 1 when the weights are the Sato-Vartia index weights.

The weights in (6) are obtained when the objective function in (6) treats all discrepancies to be of equal importance. The following result provides a solution in the weighted optimization.

Result 3: Suppose we have a set of observed weights, ϕ_i^{jk} for all *i,j and k* and we employ weights $\alpha_{ijk}\,(i = 1,2,...,N;\ j,k = 1,2,...,M)$ for each squared-deviation $\left(\phi_i^{jk} - \phi_i\right)^2$ with the property that $\sum_{j=}^{M}\sum_{k=1}^{M}\sum_{i=1}^{N}\alpha_{ijk} = 1$ Then the optimum set of weights $\hat{\phi}_i$ that minimizes the following Lagrangian function

$$\min_{\phi_i}\sum_{k=1}^{M}\sum_{j=1}^{M}\sum_{i=1}^{N}\frac{1}{2}\alpha_{ijk}\left(\phi_i^{j,k} - \phi_i\right)^2 + \lambda\left(\sum_{k=1}^{N}\phi_i - 1\right) \tag{8}$$

are given by

$$\widehat{\phi}_i = \overline{\phi}_i^{\alpha} - \frac{1}{M}\left(N\overline{\phi}_i^{\alpha} - 1\right)\frac{M}{N}$$
$$= \frac{1}{N} + \overline{\phi}_i^{\alpha} - \overline{\phi}^{\alpha} \qquad (9)$$

where

$$\overline{\phi}_i^{\alpha} = \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\alpha_{ijk}\left(\phi_i^{j,k}\right)}{M} \quad \text{and} \quad \overline{\phi}^{\alpha} = \frac{1}{N}\sum_{i=1}^{N}\overline{\phi}_i^{\alpha}$$

Alternative weighting schemes:

We may consider the following weighting schemes

$$\alpha_{ijk} = \frac{\omega_{ik} + \omega_{ij}}{2}, \quad \alpha_{ijk} = \frac{\omega_{ik} - \omega_{ij}}{\ln(\omega_{ik}/\omega_{ij})}, \alpha_{ijk} = \sqrt{\omega_{ik}\omega_{ij}}, \text{ and } \alpha_{ijk} = \ln\left(\frac{\omega_{ik}}{\omega_{ij}}\right)$$

where $w_{ij}\ (j = 1, 2, ..., M)$ is the expenditure share of *i-th* commodity in *j*-th country.

Corollary: If $\alpha_{1jk} = \alpha_{2jk} = \ldots = \alpha_{Njk} = \alpha_{jk}$ and if, further, $\sum_{j=1}^{M}\sum_{k=1}^{M}\alpha_{jk} = 1$, then the optimum weights in (9) simplify to the following weighted average of commodity-specific weights ϕ_i^{jk}, thus

$$\widehat{\phi}_i = \sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{jk}\left(\phi_i^{j,k}\right)\right] \qquad (10)$$

Approach 3: Minimum Distance from the Factor Reversal Test

Here we consider a different type of characteristicity. Since the starting point for the Sato (1976) is the factor reversal test, we explore the possibility of generating transitive multilateral indices that are close as possible to satisfying the factor reversal test. We first consider the binary case that was considered in Sato (1976).

Factor reversal test and binary comparison between j and k:

Following equations (1), a logarithmic binary price and quantity index satisfies if the weights ϕ_i^{jk} are selected so that the following equation (2) is satisfied.

$$\ln P_{jk} + \ln Q_{jk} - \ln E_{jk} = 0 \Rightarrow \sum_{i=1}^{N}\phi_i^{jk}\left(\frac{w_{ik}}{w_{ij}}\right) = 0. \qquad (11)$$

It is easy to see from this equation that there are infinitely many solutions for the unknown $\left\{\phi_i^{jk} : i = 1, 2, ..., N\right\}$ weights. We also observe that for any solution to (11), a constant multiple of the solution is also a solution to (11). We state the following results concerning the SV index.

Result 5: The SV weights in equation (3)

$$\phi_i^{jk} = \frac{\dfrac{w_{ik} - w_{ij}}{\ln w_{ik} - \ln w_{ij}}}{\sum_{i=1}^{N} \left(\dfrac{w_{ik} - w_{ij}}{\ln w_{ik} - \ln w_{ij}} \right)} \quad i = 1, 2, ..., N$$

Satisfy equation (11) and therefore logarithmic price and quantity indexes based on these weights satisfy the factor reversal test. Thus the SV index is only one of a class of log-change index numbers that satisfy the factor reversal test.

Result 6: A necessary and sufficient condition for the uniqueness (up to a scalar multiple) of the SV index is that there are only two commodities, that is N=2.

In the case of two countries and two commodities, we can solve equation (11) and show that the SV weights are unique. For N=3, we obtain several solutions which satisfy the factor reversal test.

Our result is not inconsistent with the result reported in Fattore (2009) but upon a closer examination, Fattore (2009) considers a very special structure for the observed quantities. In the absence of any such restrictions, the SV index is only one of many log-change index numbers that satisfy the factor reversal test. Thus selection of the SV index then has to be justified on other grounds, for example the SV index is exact for a CES utility function.

Factor Reversal Test and the case of multilateral comparisons:

We first observe that it is impossible to find a single set of weights $\{\phi_i : i = 1, 2, ..., N\}$ which will satisfy the factor reversal test unless the expenditure shares are identical across all the countries. Thus we observe:

$$\sum_{i=1}^{N} \left(\phi_i \cdot \ln \left(\frac{w_{ik}}{w_{ij}} \right) \right) \neq 0 \;\; for\,(j, k = 1, 2, ..., M)\; j \neq k$$

Our objective is to find a set of weights $\{\phi_i : i = 1, 2, ..., N\}$ which have minimum sum of squares of deviations from 0. Thus the problem is one of minimizing

$$\sum_{j=1}^{M} \sum_{k=1}^{M} \left(\sum_{i=1}^{N} \left(\phi_i \cdot \ln \left(\frac{w_{ik}}{w_{ij}} \right) \right) \right)^2 \; subject\ to\ 0 \leq \phi_i \leq 1\ and\ \sum_{i=1}^{N} \phi_i = 1$$

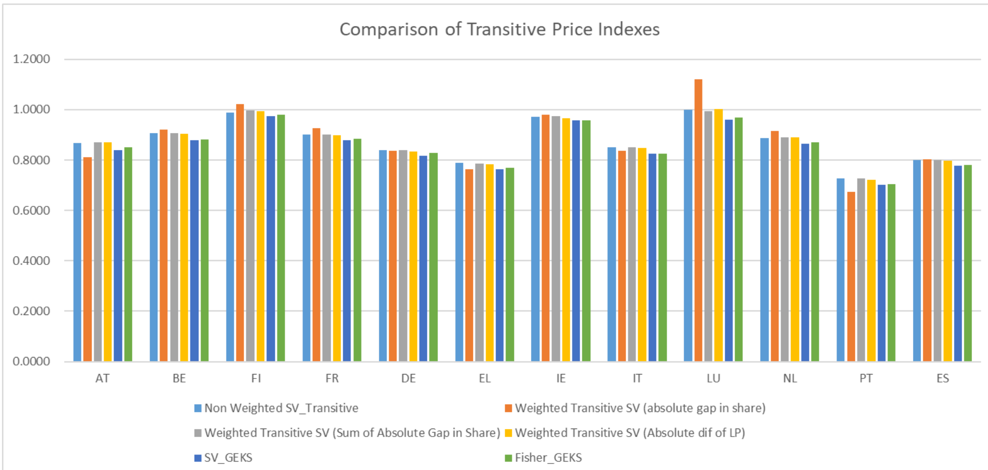
Or a weighted least squares criterion

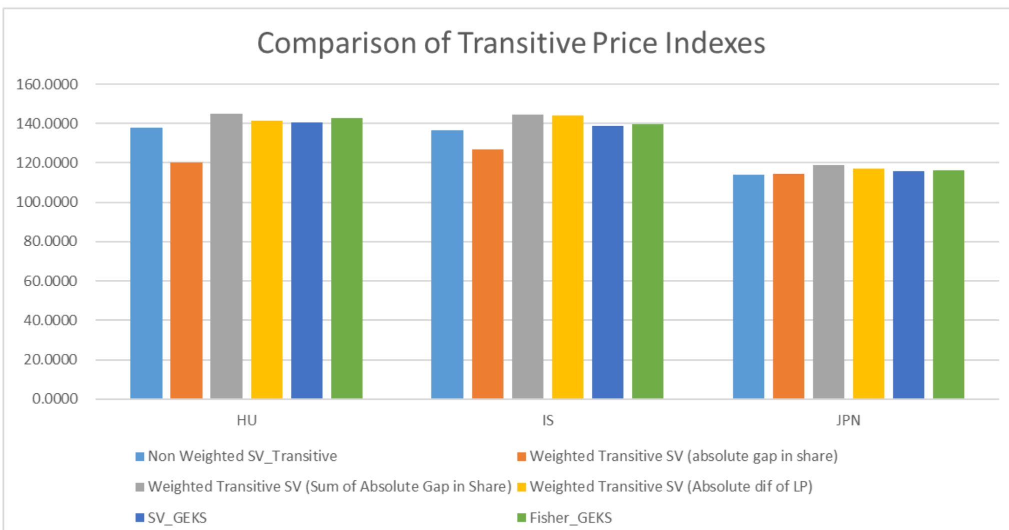
$$\sum_{j=1}^{M} \sum_{k=1}^{M} \alpha_{jk} \left(\sum_{i=1}^{N} \left(\phi_i \cdot \ln \left(\frac{w_{ik}}{w_{ij}} \right) \right) \right)^2 \; subject\ to\ 0 \leq \phi_i \leq 1\ and\ \sum_{i=1}^{N} \phi_i = 1$$

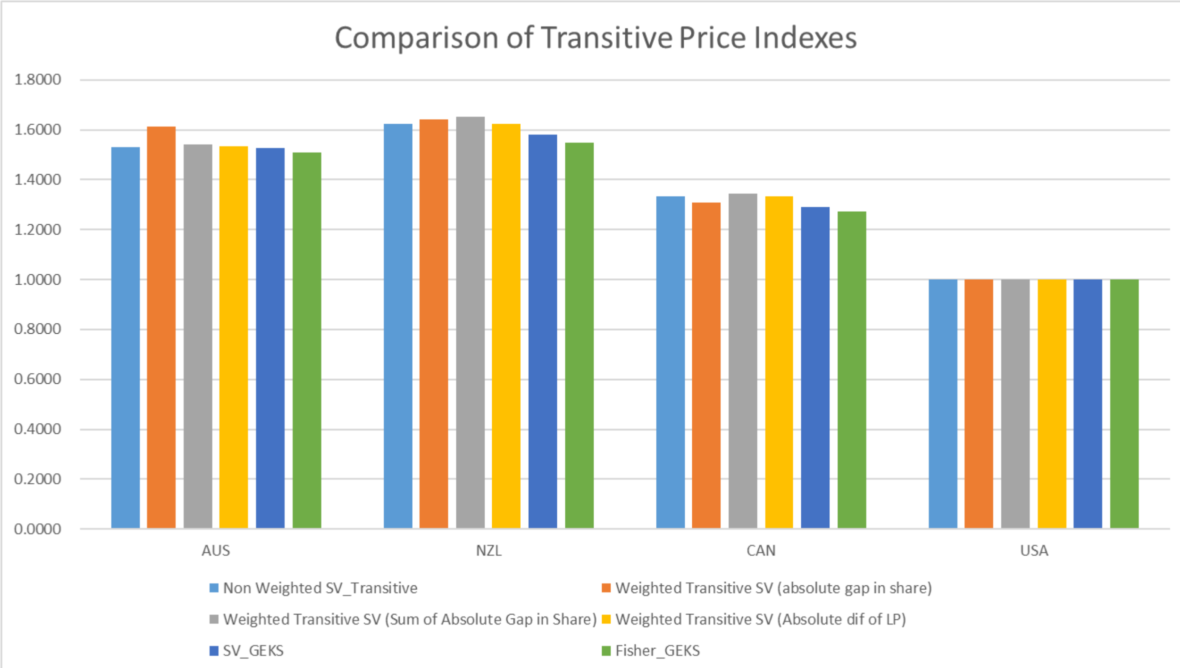
We have not obtained a satisfactory solution to this optimization problem but have results for special cases reported in Results 5 and 6.

Empirical results for 2011

We implement various versions of transitive Sato-Vartia index for multilateral comparisons using basic heading level from ICP 2011. Based on our computations for the Household Consumption, we compare the standard GEKS and our transitive Sato-Vartia Price index. The following figures show some results of comparisons, which indicates that in most countries, our transitive Sato-Vartia price indexes are similar to GEKS. However, we can notice that in some countries, there are more than 5 % discrepancies in some countries from GEKS such as New Zealand and Korea. In the charts below, we present price level indices (PPP/XR) where PPPs are computed using variants of the SV index discussed here.







The results here indicate that the SV GEKS and Fisher based GEKS are close to each other. This is to be expected as the binary SV index is usually numerically close to the Tornqvist binary index and it is well known that the Tornqvist and Fisher binary indexes are generally close to each other numerically. However weighted and unweighted transitive SV indices discussed in equations (7) and (10) seem to differ from Fisher-based GEKS. It is also clear that the choice of weights has a significant effect. Thus it is important to choose weights carefully. This is a topic that requires further work.

2. COLI for Constant Elasticity of Substitution (CES) preferences and Transitive Sato-Vartia Indexes

The Sato-Vartia price index is exact when the utility function is of the class of CES. When applying COLI to price indexes across countries, we need to assume that every country shares the same utility function, or at least, they must have similar preferences, which is very unlikely. However, to construct COLI for different preferences, first, we need to know the preference parameters for all the countries, which is also very difficult. In this section, we show that as far as we assume homothetic CES for each country and constant preference over time, it is possible to estimate country specific utility functions and COLIs across countries using official data for Eurostat from the ICP. The estimated COLIs are in general similar to the standard GEKS based on Fisher price index except for some countries such as Denmark and Australia.

For each country k we assume that preferences are represented by the Constant Elasticity of Substitution (CES) which has been standard in many applied economics studies. The function is:

$$U_{tk} = \left(\sum_{i=1}^{N} \left(a_{tk}^{i}\right)\left(q_{tk}^{i}\right)^{\frac{\sigma_k - 1}{\sigma_k}} \right)^{\frac{\sigma_k}{\sigma_k - 1}}, a_{tk}^{i} \geq 0, \sigma_k > 0. \quad (12)$$

Note that in addition to country subscript, *k*, we introduce time subscript, *t*, in the above utility function. The reason is we need time variations to estimate the demand elasticity for each country, σ_k. The unit cost function can be shown to be:

$$E_{tk}\left(p_{tk},1\right)=\left(\sum_{i=1}^{N}\left(a_{tk}^{i}\right)^{\sigma_k}\left(p_{tk}^{i}\right)^{1-\sigma_k}\right)^{\frac{1}{1-\sigma_k}} \qquad (13)$$

To make comparisons across countries, we need to impose one restriction on the parameters, a_{tk}^{i} $\left\{a_{tk}^{i}\ :\ i=1,2,...,N\right\}$. Hottman, Redding and Weinstein (2016) assume that the geometric average of a_{tk}^{i}'s is equal to 1. Instead, we choose the normalization $\sum_{i=1}^{N}a_{k}^{i}=1$. Under this normalization, it is possible to show that the CES function simplifies to the Cobb-Douglas function when $\sigma_k=1$.

In many papers that investigate the return to the variety, elasticities are assumed to be greater than unity, otherwise monopolistic firms can earn infinite profits. For example, Broda and Weinstein (2010) require σ_k to be in the range [1.05, 131.5]. Since we are not interested in the variety effects in this paper, we do not impose any restriction and in our work estimated σ_k are all positive.

Cost of Living Index (COLI) across Countries

The cost of living index between country k and j is the ratio of expenditures when both countries enjoy the same level of utility level, say U, under the price vector, p_{tk} and p_{tj}.

$$COLI\left(t,j,k\right)=\frac{E_j\left(p_{tj},U\right)}{E_k\left(p_{tk}.U\right)}$$

If the preferences of the two countries are identical, we can use some superlative indices as the approximations of the above COLI. However, if the preferences are different across countries, the standard superlative indices such as the Fisher index are no longer good approximations unless the differences can be treated as additive residuals. In general, to construct COLIs under heterogeneous preferences, we need to specify the unit cost function for each country. Note that above binary COLI is transitive, that is,

$$COLI\left(t,j,k\right)\times COLI\left(t,k,l\right)=COLI\left(t,j,l\right)\text{ for all } j,k,l$$

Also note that if the preferences of all the countries are homothetic, the unit cost functions do not depend on the level of utility, which makes the COLIs very simple. Thanks to the homothetic preferences, the indirect quantity index defined as follows is identical to the ratio of utilities in the two countries,

$$
\begin{aligned}
Q(t,j,k) &= \frac{\sum_{i=1}^{N} p_{tk}^{i} q_{tk}^{i}}{\sum_{i=1}^{N} p_{tj}^{i} q_{tj}^{i}} \frac{1}{COLI(t,j,k)} \\
&= \frac{E_j(p_{tk}, U_{kt})}{E_k(p_{tj}, U_{jt})} \frac{E_k(p_{tk})}{E_j(p_{tj})} \\
&= \frac{E_j(p_{tj}) U_{kt}}{E_k(p_{tk}) U_{jt}} \frac{E_k(p_{tk})}{E_j(p_{tj})} \\
&= \frac{U_{kt}}{U_{jt}}.
\end{aligned}
$$

Data

We make use of data for the OECD countries for the 2011 and 2014 ICP. We have basic heading level PPPs and expenditures in national currency units and hence expenditure shares. To estimate the COLIs, we need to aggregate prices over categories. Unfortunately, the price data in ICP is not suitable for aggregating price levels across categories, we need additional information to obtain the price levels in each category. These data are available from Eurostat 2012 and 2014 publications.

In the empirical application data are available only at the basic heading level, so this is the finest level of disaggregation we consider. Since the CES utility function is aggregation consistent, we consider commodity categories such as Meat as a function of the basic headings within the category (pork, chicken, lamb etc). Once the aggregate price index is obtained for each commodity category, we estimate a CES function whose elements are commodity categories.

Estimation of a_{tk}^{i}

To estimate the elasticity, We make use of the normalization $\sum_{i=1}^{N} a_{tk}^{i} = 1$. Given σ_k we can estimate $a_{tk}^{i}s$ as below. We have the expenditure share as a function:

$$
w_{tk}^{i} = \left(a_{tk}^{i}\right)^{\sigma_k} \left(\frac{p_{tk}^{i}}{P_{tk}}\right)^{1-\sigma_k}
$$

$$
\left(a_{tk}^{i}\right)^{\sigma_k} = w_{tk}^{i} \left(\frac{p_{tk}^{i}}{P_{tk}}\right)^{\sigma_k - 1}
$$

Taking the ratio relative to item 1, we have

$$\frac{\left(a_{tk}^{i}\right)^{\sigma_k}}{\left(a_{t}^{1}\right)^{\sigma_k}} = \frac{w_{tk}^{i}}{w_{tk}^{1}}\left(\frac{p_{tk}^{i}}{p_{tk}^{1}}\right)^{\sigma_k - 1}$$

$$\frac{a_{tk}^{i}}{a_{tk}^{1}} = \left(\frac{w_{tk}^{i}}{w_{tk}^{1}}\right)^{\frac{1}{\sigma_k}}\left(\frac{p_{tk}^{i}}{p_{tk}^{1}}\right)^{\frac{\sigma_k - 1}{\sigma_k}}$$

$$a_{tk}^{i} = a_{tk}^{1}\left(\frac{w_{tk}^{i}}{w_{tk}^{1}}\right)^{\frac{1}{\sigma_k}}\left(\frac{p_{tk}^{i}}{p_{tk}^{1}}\right)^{\frac{\sigma_k - 1}{\sigma_k}}$$

$$\sum_{i=2}^{N} a_{tk}^{i} = 1 - a_{tk}^{1} = a_{tk}^{1}\sum_{i=2}^{N}\left(\frac{w_{tk}^{i}}{w_{tk}^{1}}\right)^{\frac{1}{\sigma_k}}\left(\frac{p_{tk}^{i}}{p_{tk}^{1}}\right)^{\frac{\sigma_k - 1}{\sigma_k}}$$

$$a_{tk}^{1} = \frac{1}{1 + \sum_{i=2}^{N}\left(\frac{w_{tk}^{i}}{w_{tk}^{1}}\right)^{\frac{1}{\sigma_k}}\left(\frac{p_{tk}^{i}}{p_{tk}^{1}}\right)^{\frac{\sigma_k - 1}{\sigma_k}}}$$

Using this, we can obtain a_{tk}^{1} and the remaining $a_{tk}^{i}s$. If $\sigma_k = 1,$ then the resulting $a_{tk}^{i}s$ are identical to the expenditure shares.

Estimation of σ_k

In this paper, we estimate σ_k for each country and category following the method developed by Feenstra (1994). Feenstra (1994) showed that if we have a panel data of expenditure share and prices, it is possible to estimate both supply and demand elasticities consistently. The preference parameters,a\ we need a panel dataIn this paper we estimate σ for each country, each category, and for each year. For a given category, like Meat, there are several basic headings within the category. We note that under the normalization $\sum_{i=1}^{N} a_{tk}^{i} = 1.$, the demand and expenditure share functions can be written as:

$$q_{tk}^{i} = \left(a_{tk}^{i}\right)^{\sigma_k}\left(\frac{p_{tk}^{i}}{P_{tk}}\right)^{-\sigma_k} U_{tk}$$

and

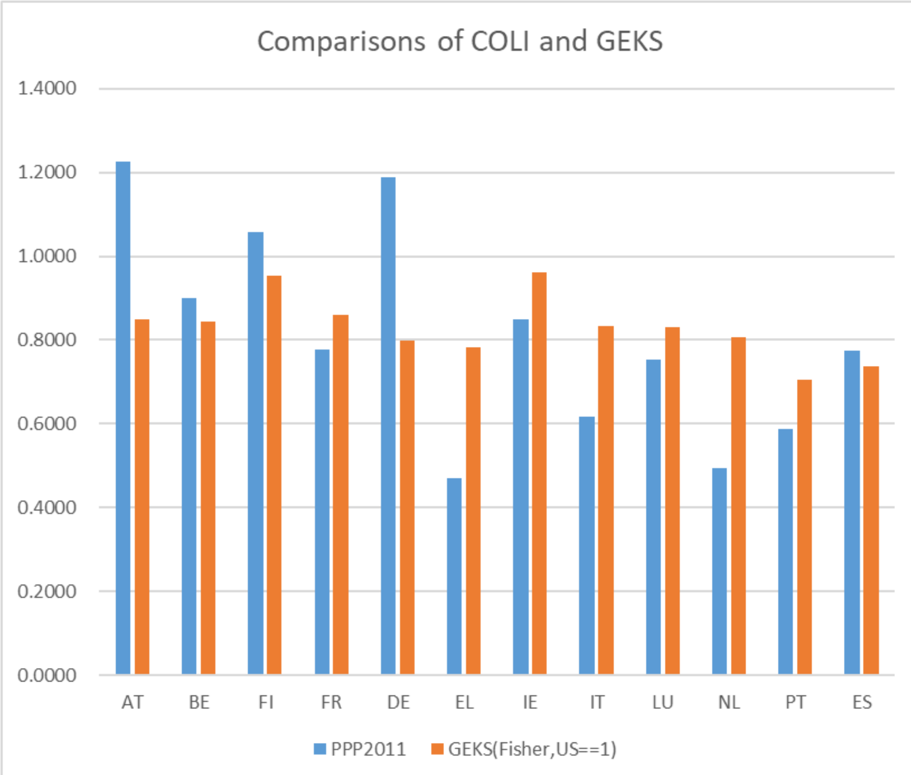
$$w_{tk}^{i} = \left(a_{tk}^{i}\right)^{\sigma_k}\left(\frac{p_{tk}^{i}}{P_{tk}}\right)$$

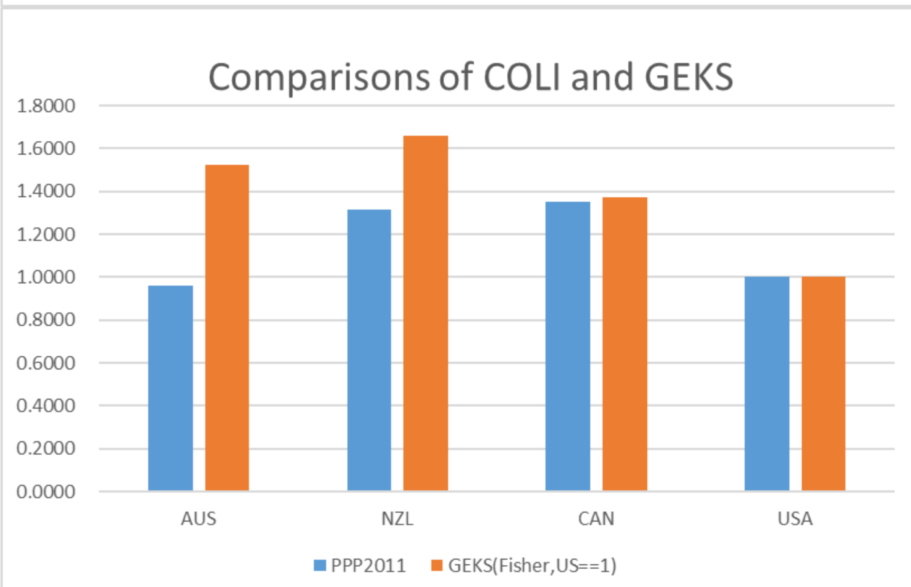
The expenditure equation forms the basis for the estimation of σ_k. We use the Feenstra (1994) method but employ the method of moments rather than using the linear approximation used in Feenstra (1994).

Elasticities are estimated twice, first at the category level and then for the whole consumption. In our estimation, we have not encountered any negative estimates of σ_k. In some cases, we had σ_k less than unity. At the category level the elasticities tended to be large with average elasticity 3.2. However, for consumption as a whole the elasticity is 1.25 very close to Cobb-Douglas.

Estimation of COLIs

Once we obtain σ_k and a_{tk}^i, it is straightforward to calculate the COLIs when preferences in all the countries are homothetic. The estimated COLI for the consumption aggregate for the year 2011, denoted as PPP2011, are presented for EU countries. These are compared and contrasted with the GEKS PPPs in the charts below.





From these charts we observe that there CES COLI based PPPs (in blue) differ significantly from the GEKS based PPPs for most countries with the exception of Canada where the results are almost the same.

The work on the estimation of COLI is work in progress and in the more complete version to be presented at the May Conference, we will have a more complete set of results.