The Imaginary Fibbing

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1 Introduction

In nature, spirals have found there way into the very genetics plant life. Spirals can be observed in flower petals, tree brand formation, and even every day fruits. There is no explanation to this phenomenon, other than it must be the most efficient formation in nature. This spiral can be approximated by the Fibonacci sequence, using each term as a length of a square, we can rearrange them into a specific way, and approximate the Golden Spiral. Within the Golden Spiral is an important number. As the Golden Spiral

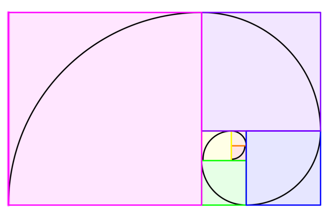


Figure 1: Approximation of Golden Spiral

spirals out, every quarter turn in it, the spiral grows by a factor of $\frac{1+\sqrt{5}}{2}$ more well known as the Golden Ratio, ϕ.

The Fibonacci Sequence can be defined as a sequence of terms, which are the sum of the two previous terms. It can be written as:

$$F_n = F_{n-1} + F_{n-2}, F_0 = 1, F_1 = 1$$

If we solve this recurrence relation in the usual way of using a characteristic equation, we find the explicit formula to be very interesting:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) = \frac{1}{\sqrt{5}}\left(\phi^n - (\frac{-1}{\phi})^n\right)$$

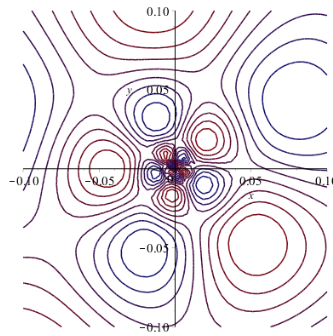


Figure 2: A Weird Looking Golden Spiral

Professor Farris posed the question to me: what if we used the same recurrence relation for Fibonacci numbers, but instead we had $F_0 = 1$ and $F_1 = i$, where $i^2 = -1$. We wonder if there is another way we can visualize the a Golden Spiral once again with this new sequence. Sadly, visualization is not so easy, imaginary numbers are a bit more fickle. Complex numbers can be visualized with a specific set of tools; especially, logarithmic spirals, and the complex exponential map. Using these ideas, we can create lines or contours for our curves, and hope to create spirals out of them using these ideas.

2 Inquiry

Investigating the new complex Fibonacci sequence, we will return to a familiar recurrence relation with new initial values:

$$G_n = G_{n-1} + G_{n-2}, \text{ for } n \geq 2$$

Here are the first few terms: $1, i, 1+i, 1+2i, 2+3i, 3+5i, 5+8i, 8+13i, \ldots$. Like the Fibonacci sequence, finding the explicit formula may be helpful in understanding or observing anything special about this specific problem. Solving in the usual way of the characteristic equation we will find that:

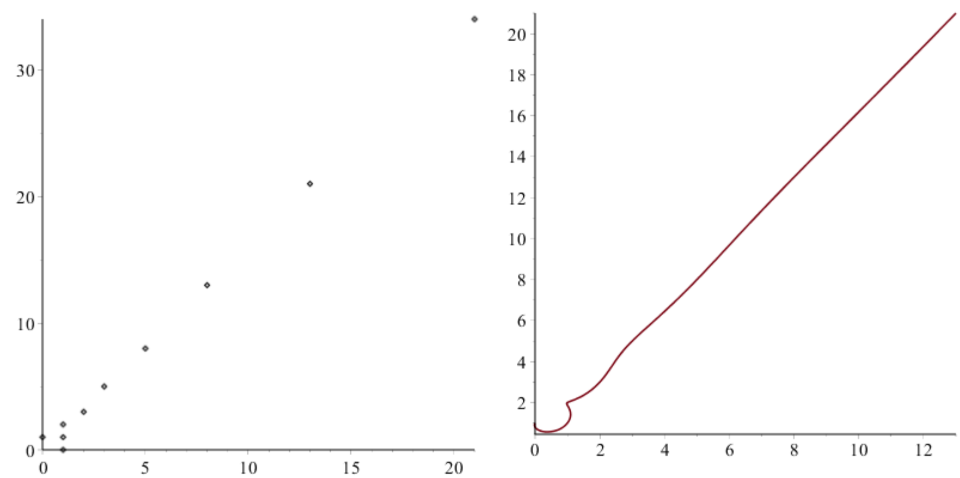
$$G_n = \left(\frac{-1+\sqrt{5}+2i}{2\sqrt{5}}\right)\left(\phi\right)^n + \left(\frac{1+\sqrt{5}-2i}{2\sqrt{5}}\right)\left(\frac{-1}{\phi}\right)^n, \text{ for any } n$$

You may notice that the real and imaginary components seem to follow an interesting pattern: $\mathrm{Re}(G_n)$: $1, 0, 1, 1, 2, 3, 5, 8, \ldots$ and $\mathrm{Im}(G_n)$:

$0, 1, 1, 2, 3, 5, 8, 13, ...,$ both seem to be very related to the Fibonacci sequence. We may now write G_n in terms of F_n:

$$G_n = F_{n-2} + F_{n-1}i, \text{ for } n \geq 2$$

In an attempt to visualize this sequence, I graphed the terms on a complex plane to see what kind of intuition might be present:



(a) points plotted

(b) points connected

Figure 3: Basic Plotting of the terms in G_n on the complex plane

This graph seems to approach a constant slope, as the terms grow successively larger. This may be due to the explicit formula for $F_n = F_n = \frac{1}{\sqrt{5}}(\phi^n - \frac{-1}{\phi}^n)$. Where if you notice, for sufficiently large n, $F_n \approx \phi^n$.

Then Professor Farris suggested that I investigate this ratio of successive terms in our G_n.

$$R_n = \frac{G_n}{G_{n+1}} : \tag{1}$$

$$R_1 = \frac{G_0}{G_1} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{(0) - i}{1} \tag{2}$$

$$R_2 = \frac{G_1}{G_2} = \frac{i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1+i}{2} \tag{3}$$

$$R_3 = \frac{G_2}{G_3} = \frac{1+i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{3-i}{5} \tag{4}$$

$$R_4 = \frac{G_3}{G_4} = \frac{1+2i}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{8+i}{13} \tag{5}$$

We then may find it worthwhile to investigate $\frac{G_n}{G_{n+1}}$ in terms of F_n, let us call it R_n.

$$R_n = \frac{G_n}{G_{n+1}} = \frac{F_{n-2} + F_{n-1}i}{F_{n-1} + F_n i}$$

Multiplying by the conjugate to remove the i from denominator as we did before, we can find that:

$$R_n = \frac{F_{n-2} + F_{n-1}i}{F_{n-1} + F_n i} \cdot \frac{F_{n-1} - F_n i}{F_{n-1} - F_n i} = \frac{(F_{n-2}F_{n-1} + F_{n-1}F_n) + (F_{n-1}^2 - F_{n-2}F_n)i}{F_{n-1}^2 + F_n^2}$$

Observing each major term, We can see that the first term in the numerator constantly grows, and looks like it gives every even numbered Fibonacci number, while the denominator gives every odd Fibonacci number, specifically in pairs F_{2n-1} and F_{2n}, respectively. And the coefficient in front of i, we can see is just a $(-1)^{n+1}$ from the terms we have already listed. As n gets sufficiently large, the imaginary part of this function drops out, and we are left with approximately:

$$\lim_{n->\infty} \frac{F_{2n-1}}{F_{2n}} \approx \frac{\phi^n}{\phi^{n+1}} \approx \frac{1}{\phi}$$

Now we have solved for the slope we saw earlier in our Figure 2. We can observe that since our G_n has the specific property of its real and imaginary coefficients being successive Fibonacci terms. We make a realization of the ratio by taking a ratio of the magnitude of the complex numbers, and whose argument is the difference in arguments of the successive terms. Approximating to the reciprocal of the familiar ϕ have been working with.

We may conjecture:

1. $(F_{n-2}F_{n-1} + F_{n-1}F_n) = F_{2n-1}$
2. $F_{n-1}^2 + F_n^2 = F_{2n}$
3. $(F_{n-1}^2 - F_{n-2}F_n) = (-1)^{n+1}$

Proof: $F_{n-2}F_{n-1} + F_{n-1}F_n = F_{2n-1}$ and $F_{n-1}^2 + F_n^2 = F_{2n}$

Initially, since these two conjectures are separate, it would make sense to solve them separately by induction. However, in the method of doing induction on recurrence relations, there will be issues, since the right side will always be decomposed into some term less than it and two less than it. Therefore, using both of the inductive hypotheses from these separate proofs in conjunction will lead to a finished proof for both.

Base Case: $n = 2$, $n = 1$ respectively for our conjunction:

$$F_0F_1 + F_1F_2 = F_3 \qquad F_0^2 + F_1^2 = F_2 \tag{6}$$

$$(1)(1) + (1)(2) = (3) \qquad (1)^2 + (1)^2 = (2) \tag{7}$$

Our conjunction would now be, suppose for some $n = k$ that$F_{k-2}F_{k-1} + F_{k-1}F_k = F_{2k-1}$ and $F_{k-1}^2 + F_k^2 = F_{2k}$ are true. Now investigate $n = k + 1$. We will do each case separately, but will use both inductive hypotheses in conjunction to prove this.

Inductive Step: for $n = k + 1$ for $F_{n-2}F_{n-1} + F_{n-1}F_n = F_{2n-1}$

$$F_n = F_{n-1} + F_{n-2}$$

$$F_n^2 + F_nF_{n-1} = F_{n-1}^2 + F_n^2 + F_{n-2}F_{n-1}$$

$$F_{n-1}F_n + F_n(F_n + F_{n-1}) = F_{n-1}^2 + F_n^2 + F_{n-2}F_{n-1} + F_{n-1}F_n$$

$$F_{n-1}F_n + F_n(F_n + F_{n-1}) = F_{2n} + F_{2n-1}$$

$$F_{n-1}F_n + F_nF_{n+1} = F_{2n+1}$$

Inductive Step: for $n = k + 1$ for $F_{n-1}^2 + F_n^2 = F_{2n}$

$$F_n = F_{n-1} + F_{n-2}$$

$$F_nF_{n-1} = F_{n-1}^2 + F_{n-2}F_{n-1}$$

$$F_nF_{n-1} + F_{n-1}^2 = 2F_{n-1}^2 + F_{n-2}F_{n-1}$$

$$F_n^2 + F_n^2 + 2F_nF_{n-1} + F_{n-1}^2 = 2(F_{n-1}^2 + F_n^2) + F_{n-2}F_{n-1} + F_{n-1}F - n$$

$$F_n^2 + (F_n + F_{n-1})^2 = 2(F_{n-1}^2 + F_n^2) + F_{n-2}F_{n-1} + F_{n-1}F_n$$

$$F_n^2 + (F_n + F_{n-1})^2 = F_{2n} + F_{2n-1} + F_{2n}$$

$$F_n^2 + F_{n+1}^2 = F_{2n+1} + F_{2n}$$

$$F_n^2 + F_{n+1}^2 = F_{2n+2}$$

Proof: $(F_{n-1}^2 - F_{n-2}F_n) = (-1)^{n+1}$

Base Case: $n = 2$

$$F_1^2 - F_2F_0 = (-1)^{2+1}$$

$$(1) - (2)(1) = (-1)^3$$

$$-1 = -1$$

Inductive Hypothesis: suppose for some $n = k$ that $(F_{k-1}^2 - F_{k-2}F_k) = (-1)^{k+1}$ is true. Inductive Step: $n = k + 1$.

$$F_k = F_{k-1} + F_{k-2}$$

$$F_k^2 - F_kF_{k-1} = F_kF_{k-2}$$
$$F_k^2 - (F_k + F_{k-1})F_{k-1} = (F_{k-1}^2 - F_kF_{k-2})(-1)$$
$$F_k^2 - F_{k+1}F_{k-1} = (-1)^{k+2}$$

These Fibonacci identities are useful, in allowing us to know that for every term in our G_n, this ratio will remain true. And that for any G_n term we want, we can trust that we will be able to make some spiral out of it.

3 The Recipe

Professor Farris' book *Creating Symmetry: The Artful Mathematics of Wallpaper Patterns*, and some inspiration from work by John Edmark, both provide strong guidance and inspiration in this project. Professor Farris' book focusing alot on wallpapers, has a strong point about explaining lattice coordinates, and how we can create them to satisfy certain function conditions. John Edmark's work provides inspiration with his art, and application of the Golden Angle. Edmark would find spiral in nature, or create his own,

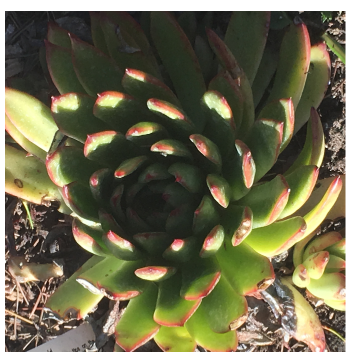


Figure 4: A succulent outside Mayer Theatre

then he would rotate them by the Golden Angle, $\frac{2\pi}{\phi^2}$. Each rotation would map one petal of the succulent to another, but be slightly further out than the original position of the petal it replaced. By creating a collection of images from these rotations, he would create this “blooming” effect on the plant. Making it seem as if it had infinite growth and petals to give. Why is that?

What we want now is to define a new coordinate system, where we can have a curve that is satisfied under certain conditions. Lattice coordinates are also a step leading into working and creating wallpapers. The most common, and nice example would be the Gaussian Integers. They are $\{a + bi \mid a, b \in \mathbb{Z}\}$, and they are made up of linear combinations of the two vectors $v_1 = 1$ and $v_2 = i$, this is the dual for the Gaussian Integers. We similarly would like to construct a lattice coordinate with our own v_1 and v_2 that satisfy some conditions we are looking for. We want a function that is invariant under a specific translation in the coordinate plane. We want something continuous, so if we are to compose Log with some f, we will need a period of $2\pi i$ to keep it continuous. some function would look like and satisfy:

$$f(z) = f\left(z + \frac{2\pi i}{G_n}\right) = f\left(z - \frac{2\pi}{G_n}\right)$$

We want a function that is invariant under these translations in the lattice coordinate system, so that we might have a nice looking curve when making these wave fronts into some desired spirals, wanting integer values is very reminiscent of an idea from Professor Farris' book, in order to make the curves connect. Professor Farris suggested that a family of functions that will satisfy these conditions would look like:

$$f(z) = e^{2\pi i(jX+kY)}$$

In order to start working with lattice coordinates, we will need some new variables from Professor Farris' book. We will begin with coordinates $[X, Y]$ for the plane by setting: $z = Xk_1 + Yk_2$. we will have our k_1 and k_2 be:

$$k_1 = \frac{2\pi i}{G_n}, k_2 = \frac{-2\pi}{G_n} = ik_1$$

And our dual, v_1 and v_2 have nice formulas, like:

$$v_1 = \frac{ik_2}{\operatorname{Im}(k_1\bar{k_2})}, v_2 = \frac{ik_1}{\operatorname{Im}(k_2\bar{k_1})}$$

And then from defining this lattice, we want k_1 and v_1 to stay out of the way of k_2 and v_2, so using the fact that we know:

$$\operatorname{Re}\left((a + bi)\overline{(c + di)}\right) = (ac + bd) = (a + bi) \cdot \overline{(c + di)}$$

So we know that taking the real component of a product of complex numbers, is the same as taking the dot product. And having "k_1 and v_1 to stay out of the way of k_2 and v_2", would be the equivalent of:

$$\begin{aligned} &\mathrm{Re}(k_1\bar{v_1}) = 1 && \mathrm{Re}(k_1\bar{v_2}) = 0 \\ &\mathrm{Re}(k_2\bar{v_1}) = 0 && \mathrm{Re}(k_2\bar{v_2}) = 1 \end{aligned}$$

It is also worth knowing that:

$$\mathrm{Re}(z) = \frac{z+\bar{z}}{2} = z + \bar{z} = 2\mathrm{Re}(z)$$

So we can set up a system of equations like:

$$\begin{aligned} k_1\bar{v_1} + \bar{k_1}v_1 = 2 &\longrightarrow k_1k_2\bar{v_1} + \bar{k_1}k_2v_1 = 2k_2 \\ k_2\bar{v_1} + \bar{k_2}v_1 = 0 &\longrightarrow k_2k_1\bar{v_1} + k_1\bar{k_2}v_1 = 0 \end{aligned}$$

And we will get that:

$$2i\mathrm{Im}(\bar{k_1}k_2)v_1 = 2k_2 \longrightarrow v_1 = \frac{ik_2}{\mathrm{Im}(k_1\bar{k_2})},$$

And similiarly for v_2:

$$v_2 = \frac{ik_1}{\mathrm{Im}(k_2\bar{k_1})}$$

From here we have formulas for our original coordinates we were looking for [X,Y], in terms of a complex number z and $\bar{v_1}$ and $\bar{v_2}$:

$$X = \mathrm{Re}(z\bar{v_1}) \tag{8}$$

$$Y = \mathrm{Re}(z\bar{v_2}) \tag{9}$$

Now that we have all the necessary pieces and our big equation for our fronts:

$$\Psi(n,x,y,j,k) = \mathrm{Re}\left(e^{2\pi i(jX(n,x,y)+kY(n,x,y))}\right)$$

The function Ψ will give us all the graphs on the left for wave fronts, using only the real part of this function in order to graph something that puts out real values. This is using the lattice coordinates we solved for earlier, and give us lines that are invariant under a translation of our k_1 and k_2. To turn these lines into spirals, we need to convert from x and y to ω, where:

$$\Psi(n,\omega(x,y),j,k) = \Psi(\mathrm{Log}(\omega)) = \Psi(\ln|\omega| + i\mathrm{Arg}(\omega))$$

$$|\omega| = \sqrt{x^2+y^2}, \mathrm{Arg}(\omega) = \arctan\left(\frac{y}{x}\right)$$

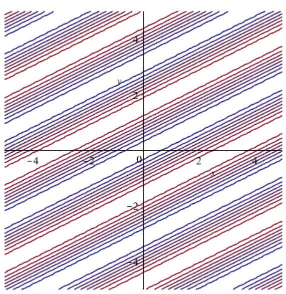
Where Arg and arctan are log-like functions functions which translate interestingly using ω.

4 Spirals

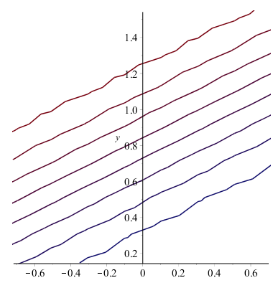
Now for assembling our desired spirals, we will will want to use the second equation, which will translate our wave fronts in the Lattice coordinates we defined into interesting spirals, if we sum and rotate the waves using $\frac{\pi}{2}$, by swapping the values of j and k, and changing their signs.

$$\Xi = \frac{1}{4}(\Psi(n, \omega, j, k) + \Psi(n, \omega, -k, j) + \Psi(n, \omega, -j, -k) + \Psi(n, \omega, -k, -j)).$$

The real parts of the given equations will give us contours. By taking the real part, we can have our function give Maple something to graph, which in turn we can make into spirals on the following pages:



(a) Contour 1



(b) Single Pack

Figure 5: $\Psi(4, x, y, 0, 1)$

In this figure we can see what our waves look like all together, and on the right is a single pack of waves. As you can see, the lines run and exist on this sort of tilted axis, if you will, created by the lattice coordinate system.

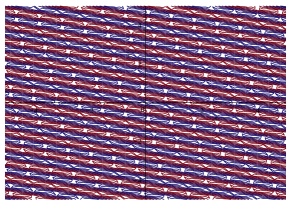
(a) Contour 2 (b) Single Pack

Figure 6: $\Psi(5, x, y, 0, 1)$

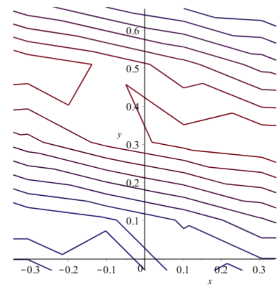
(a) Contour 3 (b) Single Pack

Figure 7: $\Psi(6, x, y, 0, 1)$

Enough of wave packets though. To create spirals, we can use the above equation with $\omega(x, y)$ to create spirals. We create and find interesting patterns and spirals by adding waves together and taking the real part each time.

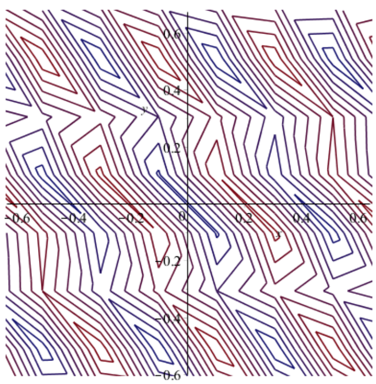


(a) Contour 4

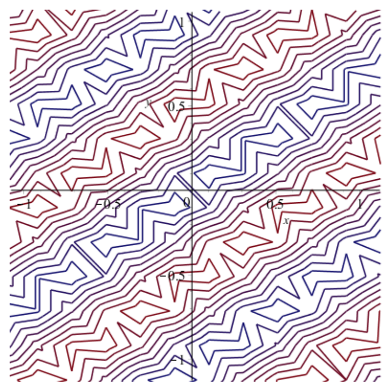


(b) Single Pack

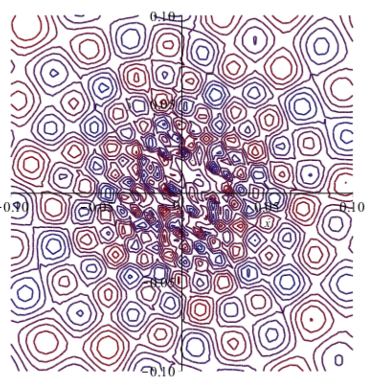
Figure 8: $\Psi(6, x, y, 1, 1)$



(a) Contour 5.1



(b) Contour 5.2



(c) Spiral 5

Figure 9: $\Xi(6, \omega(x, y), 3, 0) + \frac{1}{2}\Xi(5, \omega(x, y), 0, 2)$

5 Conclusion

We have observed a distant relative of the famous Fibonacci Sequence, and from it have created spirals from some wave contours we created using lattice coordinate systems. And translated them using a Log like function to superimpose them with one another creating spirals from a wallpaper like function. We have been able to define a family of these curves.

This family, which all produced this "blooming" effect, can be made to bloom by using a $\frac{2\pi}{\phi^2}$ rotation. If one were to wish to create an animation of these curves blooming, you would want to make a movie, with frames that are the rotation using the Golden Angle. This collection would create a blooming effect if ran in a counter clockwise way, but an "unblooming" effect if the opposite direction.

Hopefully this project can be taken further to create 3D models of these functions. Each one of these spirals is made from contours which have a value in a third variable. If one could imprint this spiral into a parabola, half sphere, or even in virtual reality, you could create some 3D objects that have this nice property of blooming, but sadly would not bloom if rotated in the normal way.