2019

AP® CollegeBoard

AP® Calculus AB

Free-Response Questions

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CALCULUS AB

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

   (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.

   (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?

   (c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

   (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.

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a) $\int_0^5 E(t)\,dt = \int_0^5 \left(20+15\sin\left(\frac{\pi t}{6}\right)\right)dt = \boxed{153 \text{ fish}}$

b) $\frac{1}{5}\int_0^5 L(t)\,dt = \boxed{6.06 \text{ fish/hour}}$

c) $F(t) = \int_0^t (E(t)-L(t))\,dt + F(0)$

$F'(t) = E(t) - L(t) = 0$

F' goes from inc to dec @ 6.204

$\boxed{t = 6.204}$

d) $R(t) = E(t) - L(t)$

$R'(5) = -10.7$

< 0

$\boxed{\text{decreasing}}$

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t (hours)	0	0.3	1.7	2.8	4
$v_P(t)$ (meters per hour)	0	55	–29	55	48

2. The velocity of a particle, P, moving along the x-axis is given by the differentiable function v_P, where $v_P(t)$ is measured in meters per hour and t is measured in hours. Selected values of $v_P(t)$ are shown in the table above. Particle P is at the origin at time $t = 0$.

(a) Justify why there must be at least one time t, for $0.3 \le t \le 2.8$, at which $v_P'(t)$, the acceleration of particle P, equals 0 meters per hour per hour.

v_p is differentiable.
By MVT, there must be some c where $v_p'(c) = \dfrac{v_p(2.8) - v_p(0.3)}{2.8 - 0.3} = 0$

(b) Use a trapezoidal sum with the three subintervals [0, 0.3], [0.3, 1.7], and [1.7, 2.8] to approximate the value of $\int_0^{2.8} v_P(t)\,dt$.

$= 0.3\left(\frac{55+0}{2}\right) + 1.4\left(\frac{55-29}{2}\right) + 1.1\left(\frac{-29+55}{2}\right)$
$= \boxed{40.75\text{m}}$

(c) A second particle, Q, also moves along the x-axis so that its velocity for $0 \le t \le 4$ is given by $v_Q(t) = 45\sqrt{t}\cos\left(0.063t^2\right)$ meters per hour. Find the time interval during which the velocity of particle Q is at least 60 meters per hour. Find the distance traveled by particle Q during the interval when the velocity of particle Q is at least 60 meters per hour.

graph $V_Q(t) - 60$
$\boxed{(1.866, 3.519)}$

$\int_{1.866}^{3.519} |V_Q(t)|\,dt = \boxed{106.1\text{m}}$

(d) At time $t = 0$, particle Q is at position $x = -90$. Using the result from part (b) and the function v_Q from part (c), approximate the distance between particles P and Q at time $t = 2.8$.

$x_Q = \int_0^{2.8} V_Q(t)\,dt - 90 = 45.9\text{m}$

$|x_P - x_Q| = |40.75 - 45.9\text{m}|$
$= \boxed{5.15\text{m}}$

END OF PART A OF SECTION II

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CALCULUS AB

SECTION II, Part B

Time—1 hour

Number of questions—4

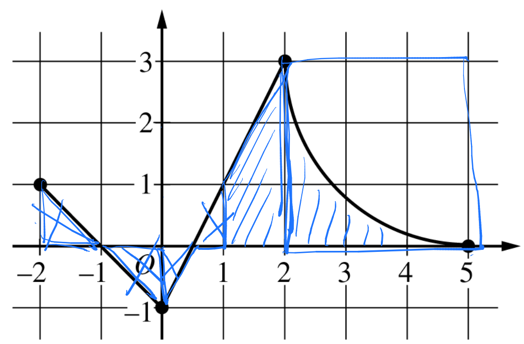
NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

$\int_{-2}^{5} f(x)\,dx$ = area

$= 1\cdot\left(\frac{1+3}{2}\right) + 3\times 3 - \frac{1}{4}\pi(3)^2$

$= 2+9-\frac{9}{4}\pi$

$= 11-\frac{9}{4}\pi$



Graph of f

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx + \int_{5}^{-2} f(x)\,dx$

$= \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx$

$= 7 - \left(11 - \frac{9}{4}\pi\right) = \boxed{-4+\frac{9}{4}\pi}$

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

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$\boxed{2+2\sqrt{5}}$

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

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$\boxed{11-\frac{9}{4}\pi}$

(d) Find $\lim_{x\to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.

$\frac{10^1 - 3f'(1)}{f(1) - \tan^{-1}(1)} = \frac{10-6}{1-\frac{\pi}{4}} = \frac{4}{\frac{4-\pi}{4}}$

$= \boxed{\frac{16}{4-\pi}}$

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b) $$\int_3^5 (2f'(x)+4)\,dx = 2\int_3^5 f'(x)\,dx + 4\int_3^5 dx$$

$$= 2\left[f(5)-f(3)\right] + 4\cdot 2$$

$$= 2\left[0-(3-\sqrt{5})\right]+8$$

$$= \boxed{2+2\sqrt{5}}$$

c) $$g(x) = \int_{-2}^{x} f(t)\,dt$$

$$g'(x) = 0 = f(x)$$

$$x = -1,\ 0.5,\ 5$$

$\uparrow$ min

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
5	$11-\frac{9}{4}\pi$

$$g(-2) = \int_{-2}^{-2} f(t)\,dt = 0$$

$$g(-1) = \int_{-2}^{-1} f(t)\,dt = \frac{1}{2}$$

$$g(5) = \int_{-2}^{5} f(t)\,dt = 11-\frac{9}{4}\pi$$

a) $V = \pi r^2 h \qquad r = 1$

$= \pi h$

$\frac{dV}{dt} = \pi \frac{dh}{dt}$

$= \pi\left(-\frac{1}{10}\sqrt{h}\right)$

$= \pi\left(-\frac{1}{10}\sqrt{4}\right)$

$= \boxed{-\frac{1}{5}\pi \ \text{ft}^3/\text{s}}$

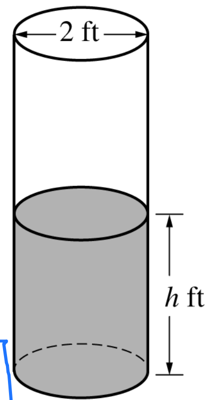
b) $\frac{dh}{dt} = -\frac{1}{10}h^{1/2}$

$\frac{d^2h}{dt^2} = -\frac{1}{20}h^{-1/2}\cdot\frac{dh}{dt}$

$= -\frac{1}{20\sqrt{h}}\cdot -\frac{1}{10}\sqrt{h}$

@ $h = 3$ $\quad \frac{d^2h}{dt^2} > 0$

so $\frac{dh}{dt}$ $\boxed{\text{increasing}}$



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$

$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$

$\int h^{-1/2}\,dh = -\frac{1}{10}t + C$

$2h^{1/2} = -\frac{1}{10}t + C$

$h = \left(-\frac{1}{20}t + C\right)^2$

$h(0) = 5 = (C)^2$

$C = \sqrt{5}$

$\boxed{h = \left(-\frac{1}{20}t + \sqrt{5}\right)^2}$

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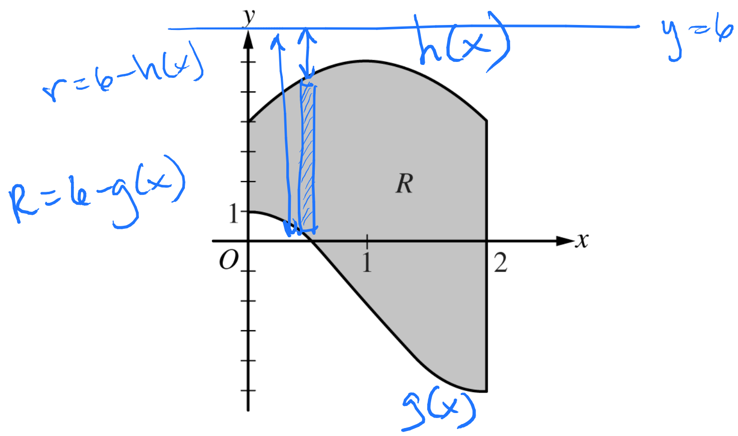

5. Let R be the region enclosed by the graphs of $g(x) = -2 + 3\cos\left(\frac{\pi}{2}x\right)$ and $h(x) = 6 - 2(x-1)^2$, the y-axis, and the vertical line $x = 2$, as shown in the figure above.

(a) Find the area of R.

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \frac{1}{x+3}$. Find the volume of the solid.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 6$.

a) $\int_0^2 (h(x)-g(x))\,dx = \int_0^2 \left[6-2(x-1)^2+2-3\cos\left(\frac{\pi}{2}x\right)\right]dx$

$= 6x - \frac{2}{3}(x-1)^3 + 2x - 3\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)\Big|_{x=0}^{2}$

$= 12 - \frac{2}{3}(1^3-(-1)^3) + 4 - 0 = 16 - \frac{4}{3} = \boxed{\frac{44}{3}}$

b) $\int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx = \ln x+3 \Big|_0^2 = \ln 5 - \ln 3$
$= \boxed{\ln\frac{5}{3}}$

c) $\boxed{\int_0^2 \pi\left[(6-g(x))^2-(6-h(x))^2\right]dx}$

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6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

   (a) Find $h'(2)$. $\boxed{h'(2) = \frac{2}{3}}$ tangent line at $x=2$ has a slope of $\frac{2}{3}$

   (b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

   (c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

   (d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer. $\boxed{\text{yes}}$

   see next page

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b) $a'(x) = 3\left[3x^2h(x) + x^3h'(x)\right]$

$a'(2) = 3\left(3(2)^2h(2) + 2^3h'(2)\right) = 3\left(\underbrace{12\cdot 4}_{48} + \underbrace{8\cdot\frac{2}{3}}_{\frac{16}{3}}\right) = 144 + 16$

$= \boxed{160}$

STOP

END OF EXAM

c) $\lim_{x\to 2} \dfrac{x^2-4}{1-(f(x))^3} = \dfrac{0}{0}$ to use LH

so $1 - f(2)^3 = 0$ so $\boxed{f(2) = 1}$

using LH $= \lim_{x\to 2} \dfrac{2x}{-3f(x)^2f'(x)} = \dfrac{2(2)}{-3(1)^2f'(2)} = h(2)$ h is continuous

$= 4$

$\dfrac{4}{-3f'(2)} = 4$

$\boxed{f'(2) = -\frac{1}{3}}$

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d) $g(2) \le k(2) \le h(2)$

$4 \le k(2) \le 4$ so $k(2) = 4$

$\lim_{x\to 2} g(x) = 4$ and $\lim_{x\to 2} h(x) = 4$

b/c g, h are continuous (differentiable)

by squeeze theorem

$$\lim_{x\to 2} g(x) \le \lim_{x\to 2} k(x) \le \lim_{x\to 2} h(x)$$

$$\lim_{x\to 2} k(x) = 4 = k(2)$$

so it is continuous @ $x = 2$