2019

AP® | CollegeBoard

AP® Calculus BC

Free-Response Questions

CALCULUS AB

SECTION II, Part A

Time—30 minutes

Number of questions—2

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

1. Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight ($t = 0$).

   (a) How many fish enter the lake over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)? Give your answer to the nearest whole number.

   (b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight ($t = 0$) to 5 A.M. ($t = 5$)?

   (c) At what time t, for $0 \le t \le 8$, is the greatest number of fish in the lake? Justify your answer.

   (d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.

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a) $\int_0^5 E(t)\,dt = \int_0^5 \left(20 + 15\sin\left(\frac{\pi t}{6}\right)\right)dt = \boxed{153 \text{ fish}}$

b) $\frac{1}{5}\int_0^5 L(t)\,dt = \boxed{6.06 \text{ fish/hour}}$

c) $F(t) = \int_0^t (E(t) - L(t))\,dt + F(0)$

$F'(t) = E(t) - L(t) = 0$

$\boxed{t = 6.204}$

F' goes from inc to dec @ 6.204

d) $R(t) = E(t) - L(t)$

$R'(5) = -10.7 < 0$

$\boxed{\text{decreasing}}$

a) $A = \int \frac{1}{2} r^2 \, d\theta$

$= \int_0^{\sqrt{\pi}} \frac{1}{2} \left[3\sqrt{\theta} \sin \theta^2 \right]^2 d\theta$

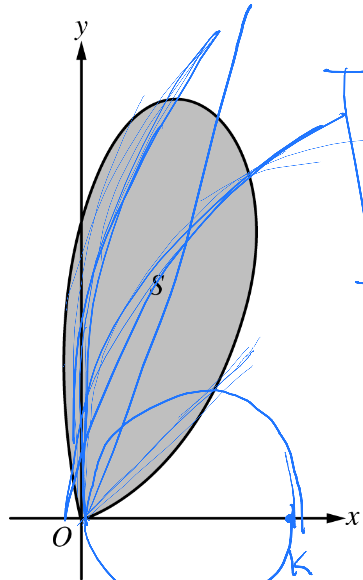
$= \boxed{3.534}$

b) $d = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} r \, d\theta$

$= \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} 3\sqrt{\theta} \sin \theta^2 \, d\theta$

$= \boxed{1.58}$

$\tan\theta = m$



c) $\int_0^{\tan^{-1} m} \frac{1}{2} r^2 \, d\theta = \int_{\tan^{-1} m}^{\sqrt{\pi}} \frac{1}{2} r^2 \, d\theta$

$$\boxed{\int_0^{\tan^{-1} m} 3\theta \sin^2 \theta^2 \, d\theta = \int_{\tan^{-1} m}^{\sqrt{\pi}} 3\theta \sin^2 \theta^2 \, d\theta}$$

2. Let S be the region bounded by the graph of the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$, as shown in the figure above.

(a) Find the area of S.

(b) What is the average distance from the origin to a point on the polar curve $r(\theta) = 3\sqrt{\theta}\sin\left(\theta^2\right)$ for $0 \le \theta \le \sqrt{\pi}$?

(c) There is a line through the origin with positive slope m that divides the region S into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of m.

(d) For $k > 0$, let $A(k)$ be the area of the portion of region S that is also inside the circle $r = k \cos \theta$. Find $\lim\limits_{k \to \infty} A(k)$.

as $k \to \infty$, the circle will cover all of S with $0 \le \theta \le \frac{\pi}{2}$

$\lim\limits_{k \to \infty} A(k) = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \boxed{3.324}$

END OF PART A OF SECTION II

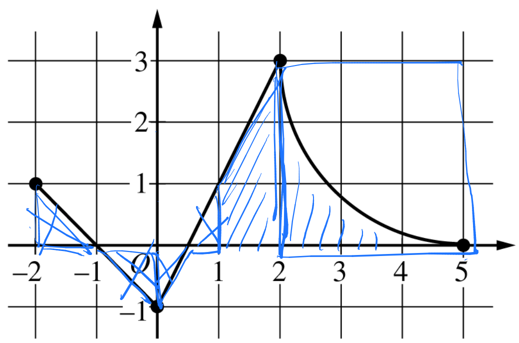
CALCULUS BC

SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of f

$\int_{-2}^{5} f(x)\,dx$ = area

$= 1\cdot\left(\frac{1+3}{2}\right) + 3\times 3 - \frac{1}{4}\pi(3)^2$

$= 2+9-\frac{9}{4}\pi$

$= 11-\frac{9}{4}\pi$

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$\int_{-6}^{-2} f(x)dx = \int_{-6}^{5} f(x)dx + \int_{5}^{-2} f(x)dx$

$= \int_{-6}^{5} f(x)dx - \int_{-2}^{5} f(x)dx$

$= 7 - \left(11-\frac{9}{4}\pi\right) = \boxed{-4+\frac{9}{4}\pi}$

(b) Evaluate $\int_{3}^{5} (2f'(x)+4)\,dx$. See next page

$\boxed{2+2\sqrt{5}}$

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer. See next page

$\boxed{11-\frac{9}{4}\pi}$

(d) Find $\lim_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

$\dfrac{10^1 - 3f'(1)}{f(1)-\tan^{-1}(1)} = \dfrac{10-6}{1-\frac{\pi}{4}} = \dfrac{4}{\frac{4-\pi}{4}}$

$= \boxed{\dfrac{16}{4-\pi}}$

b) $\int_3^5 (2f'(x)+4)\,dx = 2\int_3^5 f'(x)\,dx + 4\int_3^5 dx$

$= 2[f(5)-f(3)] + 4\cdot 2$

$= 2[0-(3-\sqrt{5})]+8$

$= \boxed{2+2\sqrt{5}}$

c) $g(x) = \int_{-2}^{x} f(t)\,dt$

$g'(x) = 0 = f(x)$

$x = -1, 0.5, 5$

↑ min

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
5	$11-\frac{9}{4}\pi$

$g(-2) = \int_{-2}^{-2} f(t)\,dt = 0$

$g(-1) = \int_{-2}^{-1} f(t)\,dt = \frac{1}{2}$

$g(5) = \int_{-2}^{5} f(t)\,dt = 11 - \frac{9}{4}\pi$

a) $V = \pi r^2 h \qquad r = 1$

$= \pi h$

$\frac{dV}{dt} = \pi \frac{dh}{dt}$

$= \pi\left(-\frac{1}{10}\sqrt{h}\right)$

$= \pi\left(-\frac{1}{10}\sqrt{4}\right)$

$= \boxed{-\frac{1}{5}\pi \ \text{ft}^3/\text{s}}$

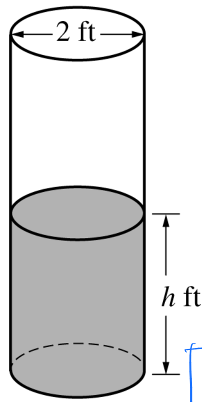
b) $\frac{dh}{dt} = -\frac{1}{10} h^{1/2}$

$\frac{d^2h}{dt^2} = -\frac{1}{20}\frac{1}{\sqrt{h}}\frac{dh}{dt}$

$= -\frac{1}{20}\frac{1}{\cancel{\sqrt{h}}}\left(-\frac{1}{10}\cancel{\sqrt{h}}\right)$

$= \frac{1}{200} > 0$

$\boxed{\frac{dh}{dt} \text{ increasing}}$



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

$\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$

$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$

$\int h^{-1/2}\,dh = -\frac{1}{10}t + C$

$2h^{1/2} = -\frac{1}{10}t + C$

$h = \left(-\frac{1}{20}t + c\right)^2$

$h(0) = 5 = (c)^2$

$c = \sqrt{5}$

$\boxed{h = \left(-\frac{1}{20}t + \sqrt{5}\right)^2}$

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

see next page $\boxed{\text{diverges}}$

a) $f'(x) = -1(x^2-2x+k)^{-2}(2x-2)$

$= -\dfrac{2x-2}{(x^2-2x+k)^2}$

$f'(0) = 6 = -\dfrac{2(0)-2}{(0^2-2(0)+k)^2} = \dfrac{2}{k^2}$

$k^2 = \dfrac{1}{3}$

$k = \sqrt{\dfrac{1}{3}} = \dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \boxed{\dfrac{\sqrt{3}}{3}}$

b) $\int_0^1 f(x)dx = \int_0^1 \dfrac{1}{x^2-2x-8} =$

$\dfrac{1}{x^2-2x-8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2} = \dfrac{1}{6}\dfrac{1}{x-4} - \dfrac{1}{6}\dfrac{1}{x+2}$

$1 = A(x+2) + B(x-4)$

$= x(A+B) + (2A-4B)$

$A+B=0 \qquad 2A-4B=1$

$B=-A \qquad 6A=1$

$A = \frac{1}{6} \quad B = -\frac{1}{6}$

$\int_0^1 f(x)dx = \frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2|$

$= \frac{1}{6}\ln\left|\dfrac{x-4}{x+2}\right| \Big|_{x=0}^{1}$

$= \frac{1}{6}\ln\left|\dfrac{-3}{3}\right| - \frac{1}{6}\ln\left|\dfrac{-4}{2}\right|$

$= \frac{1}{6}\ln 1 - \frac{1}{6}\ln 2$

$= -\frac{1}{6}\ln 2$

c) $\int_0^1 \frac{1}{x^2-2x+1}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx$

$= \lim_{c \to 1^-} \int_0^c \frac{1}{(x-1)^2}\,dx = \lim_{c \to 1^-} \left. \frac{-1}{x-1} \right|_{x=0}^{c}$

$= \lim_{c \to 1^-} \left(\frac{-1}{c-1} - \frac{-1}{0-1} \right) = \lim_{c \to 1^-} \left(\frac{-1}{c-1} - 1 \right)$

$= \lim_{c \to 1^-} \left(\frac{-1-(c-1)}{c-1} \right) = \lim_{c \to 1^-} \frac{-c}{c-1} = +\infty$

we have shown $\int_0^1 \frac{1}{x^2-2x+1}\,dx = +\infty$

now look at $\int_1^2 \frac{1}{x^2-2x+2}\,dx$

$= \lim_{c \to 1^+} \int_c^2 \frac{1}{(x-1)^2}\,dx$

$= \lim_{c \to 1^+} \left. -\frac{1}{x-1} \right|_{x=c}^{2}$

$= \lim_{c \to 1^+} -\left(\frac{1}{2-1} - \frac{1}{c-1} \right)$

$= \lim_{c \to 1^+} -\left(1 - \frac{1}{c-1} \right)$

$= \lim_{c \to 1^+} -1 + \frac{1}{c-1} = +\infty$

because both are $+\infty$, the total $\int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx = +\infty$

$\boxed{\text{diverges}}$

a) $T_3(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3$

$= \boxed{3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3}$

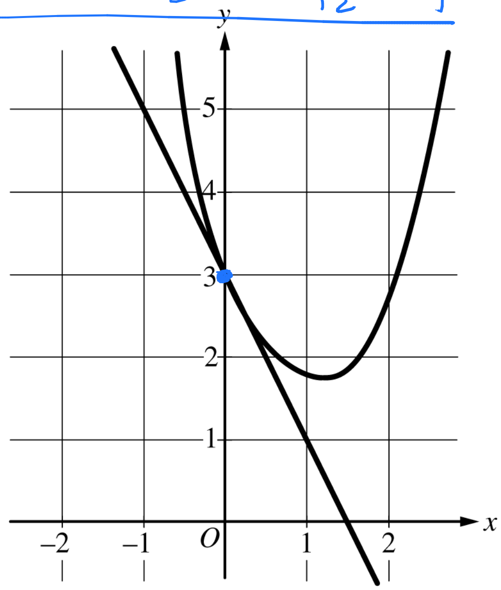
$f(0) = 3$

$f'(0) = -2$

$f''(0) = 3$

$f'''(0) = -\frac{23}{2}$

See next page for b-d



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

$T_3(x) = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

$e^x = 1 + x + \frac{1}{2}x^2$ $\qquad$ $e^x f(x) = 3 + x + x^2$

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

$\boxed{h(1) = \frac{97}{48}}$

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

$\frac{9}{20} = \boxed{0.45}$

STOP

END OF EXAM

b) $e^x \approx 1 + x + \frac{1}{2}x^2$

$$f(x)e^x \approx \left(1 + x + \tfrac{1}{2}x^2\right)\left(3 - 2x + \tfrac{3}{2}x^2\right)$$

$$= 3 + x(3-2) + x^2\left(\tfrac{3}{2} - 2 + \tfrac{3}{2}\right)$$

$$= \boxed{3 + x + x^2}$$

c) $$h(x) = \int_0^x f(t)\,dt = \int_0^x \left(3 - 2t + \tfrac{3}{2}t^2 - \tfrac{23}{12}t^3\right)dt$$

$$= 3x - x^2 + \tfrac{1}{2}x^3 - \tfrac{23}{48}x^4$$

$$h(1) = 3 - 1 + \tfrac{1}{2} - \tfrac{23}{48} = \tfrac{5}{2} - \tfrac{23}{48} = \tfrac{120 - 23}{48}$$

$$= \boxed{\tfrac{97}{48}}$$

d) error bound = next term in $h(x)$

next term in $f(x)$ is $\frac{1}{24} f^{(4)}(0)\, x^4$

$$= \tfrac{1}{24} 54 x^4 = \tfrac{9}{4} x^4$$

so next term in $h(x) = \int_0^x f(t)\,dt$

is $\frac{9}{20} x^5$

at $x = 1$, this term is $\boxed{\frac{9}{20}}$