3. Applications of Derivatives

3.1 Absolute and Relative Extrema (First Derivative Test)

Find the absolute maximum and minimum value of the function on the given interval.

1. $f(x) = x^2 + x + 4$ on $[-1, 2]$
2. $f(x) = 3\sin x$ on $[\pi/4, 2\pi/3]$
3. $f(x) = x^2\sqrt{8 - 2x^2}$ on $[-4, 4]$
4. $f(x) = x + \dfrac{3}{x}$ on $[1, 5]$
5. $f(x) = e^x \cos x$ on $[0, \pi]$
6. $f(x) = \dfrac{\ln x}{x}$ on $[1, 4]$

In the following problems, do the following:

(a) Give the domain for *f*
(b) Find the critical numbers for *f*
(c) Determine when *f* is increasing or decreasing
(d) Classify each critical number as a relative minimum, relative maximum, or neither.

7. $f(x) = x^2 + 2x - 3$
8. $f(x) = 2x^3 + x^2 - x + 3$
9. $f(x) = \dfrac{1}{x^2 - 2x + 2}$
10. $f(x) = \dfrac{x}{x^2 - 2x - 8}$
11. $f(x) = \sin x \cos x$ on $(-\pi, \pi)$

Challenge Questions

12. Find the absolute minimum and maximum of the function $f(x) = x^2 + x - 2|x|$ over the interval [-2,2]. Find and identify the local minimum and maximum.

3.2 Rolle's Theorem and Mean Value Theorem

A function *f(x)* and interval [*a*,*b*] are given. Check if Rolle's Theorem can be applied, and if so, find *c* in [*a*,*b*] such that *f'(c)* = 0.

1. $f(x) = 6$ on $[-1, 1]$
2. $f(x) = x^2 + x - 6$ on $[-3, 2]$
3. $f(x) = x^2 + x$ on $[-2, 2]$
4. $f(x) = \cos x$ on $[0, \pi]$

A function *f(x)* and interval [*a*,*b*] are given. Check if the Mean Value Theorem can be applied, and if so, find *c* in [*a*,*b*] guaranteed by the Mean Value Theorem.

5. $f(x) = x^2 + 3x - 1$ on $[-2, 2]$
6. $f(x) = \sqrt{9 - x^2}$ on $[0, 3]$
7. $f(x) = \dfrac{x^2 - 9}{x^2 - 1}$ on $[0, 2]$
8. $f(x) = \tan x$ on $[-\pi/4, \pi/4]$

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9. $f(x) = 2x^3 - 5x^2 + 6x + 1$ on $[-5, 2]$

3.3 Concavity and The Second Derivative Test

For the following functions,

a) Find the points of inflection

b) Find where the function is concave up and concave down

1. $f(x) = x^2 - 2x + 1$
2. $f(x) = x^3 - x + 1$
3. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$
4. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$
5. $f(x) = \dfrac{1}{x^2 + 1}$
6. $f(x) = x^2 \ln x$

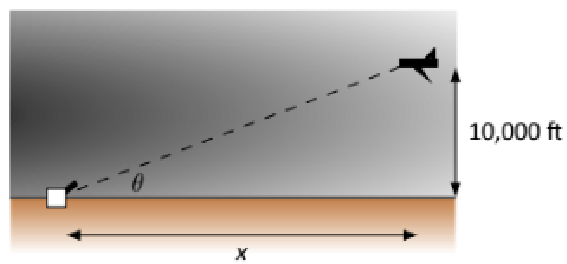
Find the critical numbers for the following functions and use the second derivative test to classify it as a minimum, maximum, or neither.

7. $f(x) = x^2 - 2x + 1$
8. $f(x) = x^3 - x + 1$
9. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$
10. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$
11. $f(x) = \dfrac{1}{x^2 + 1}$
12. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$
13. $f(x) = x^2 \ln x$
14. $f(x) = \dfrac{a}{x^2 + b^2}$
15. $f(x) = \sin(ax + b)$

3.4 Related Rates

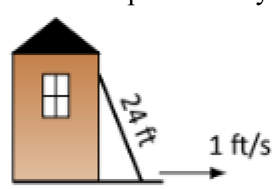
1. Water flows onto a flat surface at a rate of 5 cm^3/s forming a circular puddle 10 mm deep. How fast is the radius growing when the radius is 10 cm?

2. An F-22 aircraft is flying at 500 mph with an elevation of 10,000 ft on a straight-line path that will take it directly over an anti-aircraft gun.



How fast must the gun be able to turn to accurately track the aircraft when the plane is 1 mile away?

3. A 24 ft ladder is leaning against a house while the base is pulled away at a constant rate of 1 ft/s.



At what rate is the top of the ladder sliding down the slide of the house when the base is 10 feet from the house?

4. An inverted cylindrical cone, 20 ft deep and 10 ft across at the top, is being filled with water at a rate of 10 ft^3/min. At what rate is the water rising in the tank when the depth of the water is 10 feet? How long will the tank take to fill when starting at empty?

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5. A spherical tank with radius 10 m is losing water at a rate of 5 ft^3/min. What is the rate of change of the height of the water when the tank is half-full?

Challenge Questions

6. A clock has a 4-inch minute hand and 2-inch hour hand. When the clock shows 3 o'clock, what is the rate of change of the distance between the hour and minute hand. Note, the hour hand and the minute hand are always moving.

7. A particle is moving along $y = x^4$ with $dx/dt = 3$. Let θ bet the angle between the x-axis and the ray from the origin to the particle. Find $d\theta/dt$ at (1,1).

3.5 Optimization

1. Find the maximum product of two numbers (not necessarily integers) that have a sum of 100.

2. Find the maximum sum of two positive numbers whose product is 500.

3. Find the maximal area of a right triangle with hypotenuse of length 5.

4. A standard soda can is roughly cylindrical and holds 355 cm^3 of liquid. What dimensions should the cylinder be to minimize the material needed to produce the can?

5. The woman throws a stick into a lake for her dog to fetch; the stick is 20 feet down the shore line and 15 feet into the water from there. The dog may jump directly into the water and swim, or run along the shore line to get closer to the stick before swimming. The dog runs about 22 ft/s and swims about 1.5 ft/s.

   How far along the shore should the dog run to minimize the time it takes to get to the stick?

6. What are the dimensions of the rectangle with largest area that can be drawn inside the unit circle?

Challenge Questions

7. You are given a single piece of wire with length 10 cm, and you cut it into two pieces of wire and shaped to make a circle and an equilateral triangle. What are the dimensions of such a circle and triangle that would maximize the total area of both the triangle and circle?

8. Find the maximum sized rectangle that would fit inside of the ellipse with equation $x^2 + 9y^2 = 9$.

9. Find the point on the line $y = x^2$ that is closest to the point (0,1).

3.6 Linear Approximation and Differentials

1. Given $P(100) = -67$ and $P(100) = 5$, approximate P(110).

2. Knowing $f(1) = 25$ and $f'(10) = 5$, which approximation (using linear approximation) is likely to be most accurate: $f(10.1)$, $f(11)$, or $f(20)$? Explain your reasoning?

3. Given $H(0) = 17$ and $H(2) = 29$, approximate $H'(2)$.

4. Let v(t) measure the velocity, in ft/s, of a car moving in a straight line t sections after starting. What are the units of v'(t)?

5. Given that $e^0 = 1$, approximate the value of $e^{0.1}$ using the tangent line to $f(x) = e^x$ at $x = 0$.

6. Approximate $\sqrt[3]{1005}$ using a linear approximation around $x = 1000$.

Use differentials to approximate the following values.

7. 3.1^2

8. 4.95^2

9. $\sqrt[3]{26}$

10. $\sqrt[4]{68}$

11. sin(2)

12. ln(1.2)

Compute the differential *dy*

13. $y = x^2 + 3x - 5$

14. $y = \dfrac{1}{4x^2}$

15. $y = x^2 e^{3x}$

16. $y = \dfrac{2x}{\tan x + 1}$

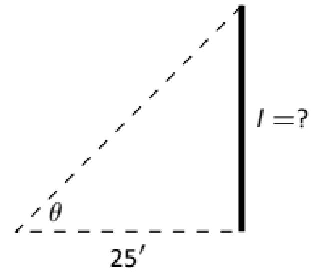
17. $y = e^x \sin x$

18. $y = x \ln x - x$

19. A set of plastic spheres are to be made with a diameter of 1 cm. If the manufacturing process is accurate to 1 mm, what is the propagated error in volume of the spheres?

20. What is the propagated error in the measurement of the cross sectional area of a circular log if the diameter is measured at 15”, accurate to ¼”?

21. The length of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85.2°, accurate to 1°. Assume that the triangle formed is a right triangle?



a) What is the measured length *l* of the wall?
b) What is the propagated error?
c) What is the percent error?

3.7 L’Hospital’s Rule

Use L’Hospital’s Rule to compute the following limits.

1. $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x - 1}$

2. $\lim\limits_{x \to \pi} \dfrac{\sin x}{x - \pi}$

3. $\lim\limits_{x \to 0} \dfrac{\sin(5x)}{x}$

4. $\lim\limits_{x \to 0} \dfrac{\sin(2x)}{\sin(3x)}$

5. $\lim\limits_{x \to 0^+} \dfrac{e^x - 1}{x^2}$

6. $\lim\limits_{x \to 0^+} \dfrac{x - \sin x}{x^3 - x^2}$

7. $\lim\limits_{x \to \infty} \dfrac{\sqrt{x}}{e^x}$

8. $\lim\limits_{x \to \infty} \dfrac{e^x}{3^x}$

9. $\lim\limits_{x \to -2} \dfrac{x^3 + 4x^2 + 4x}{x^3 + 7x^2 + 16x + 12}$

10. $\lim_{x \to \infty} \frac{\ln(x^2)}{x}$

11. $\lim_{x \to 0^+} x \cdot \ln x$

12. $\lim_{x \to 0^+} x e^{1/x}$

13. $\lim_{x \to \infty} \sqrt{x} - \ln x$

14. $\lim_{x \to 0^+} \frac{1}{x^2} e^{-1/x}$

15. $\lim_{x \to \pi/2} \tan x \sin(2x)$

16. $\lim_{x \to 3^+} \frac{5}{x^2 - 9} - \frac{x}{x - 3}$

17. $\lim_{x \to \infty} \frac{(\ln x)^3}{x}$

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